Group Assignment - Suggested Solutions

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General Comments

- Broadly speaking, I think everyone did well on their assignments
- I saw some problems computationally with **constraints that don't exist** in the question
- Case in point, investment is not restricted to $I_t \geq 0$ in any of the questions. **Inequality constraints** are easy to implement numerically, but harder to solve algebraically so aren't part of the course.
- I have no problem per se with AI/LLM-assisted coding, I use it myself and it boosts my productivity a lot. But I can spot errors. Make sure you can verify its output and bridge what you see with the question you are asked! Some subquestions are asked deliberately to get you to check that your results make sense and pass a "smell test".
- **recursive notation**: no need to keep $t, t+1$ subscripts, that whole point is the problem is timeless in certain respects. Use K' not K_{t+1} if you are writing the recursive problem.
- Commentary on findings: some comments along the lines marginal cost equals marginal benefit were a bit thin. Tell me **where the benefits to the firm come from**, and the sources of costs, and where we see these things in the mathematics.
- Personally I like to keep my **constraints explicit**, and more choice variables means the FOCs are clearer, and multipliers have economic meaning. With a substituted constraint this information is partially obscured, and risks orthographic mistakes in the written exam. Sometimes going slow (all constraints) is faster. Of course for coding the logic is completely upside-down, we want to simplyify and speed up as much as possible.
- Be careful with constraint substitution and which variables **remain as choice variables.**

Tobin

1A.1 recursive problem

Let x denote capex spending (investment). The firm's Bellman equation, subject to the law of motion of capital (with no restrictions on the sign of investment, so $x < 0$ is feasible) is:

$$\boxed{\begin{aligned} V(z,k) &= \max_{x,k'} \left\{ \pi(k) - x - \frac{\gamma}{2}\left(\frac{x}{k}\right)^2 k + \beta \sum_{z'|z} P(z'|z) V(z',k'|z,k) \right\} \quad (1) \\ s.t. \quad & k' = (1-\delta)k + x \quad (2) \end{aligned}}$$

Note how I am using the conditional expectation operator $E_{z'|z}$ and probability-weighted-sum interchangeably. Both are acceptable in the exam. The Lagrangian is then:

$$\boxed{\mathcal{L}(z,k,\lambda) = \pi(k) - x - \frac{\gamma}{2}\left(\frac{x}{k}\right)^2 k + \beta E_{z'|z} V(z',k') + \lambda\Big(x + (1-\delta)k - k'\Big)} \tag{3}$$

First order conditions:

$$[FOC/x]: \quad \lambda = (1 + \frac{\gamma x}{k}) \tag{4}$$

$$[k']: \quad \lambda = \ \beta E_{z'|z} \frac{\partial V(z',k')}{\partial k'}, \quad \text{using the envelope condition:} \tag{5}$$

$$[EC]: \quad \frac{\partial V(z,k)}{\partial k} = \alpha z k^{\alpha-1} + \frac{\gamma}{2}\left(\frac{x}{k}\right)^2 + \lambda(1-\delta) = \pi_k + \mathcal{AC}_k + \lambda(1-\delta) \tag{6}$$

$$[ST]: \quad \frac{\partial V(z',k')}{\partial k'} = \pi'_{k'} + \mathcal{AC}'_{k'} + \lambda'(1-\delta) \tag{7}$$

$$[k']: \quad \lambda = \beta E_{z'|z} \pi'_{k'} + \mathcal{AC}'_{k'} + \lambda'(1-\delta) \tag{8}$$

$$\boxed{(1 + \frac{\gamma x}{k}) = \beta E_{z'|z} \left(\pi'_{k'} + \mathcal{AC}'_{k'} + (1-\delta)(1 + \frac{\gamma x'}{k'}) \right)} \tag{9}$$

Plugging the functional forms for their expressions:

$$\boxed{(1 + \frac{\gamma x}{k}) = \beta E_{z'|z} \left(\alpha z k^{\alpha-1} + \frac{\gamma}{2}\left(\frac{x}{k}\right)^2 + (1-\delta)(1 + \frac{\gamma x'}{k'}) \right)} \tag{10}$$

Combining everything, rename the lagrange multiplier as q.

$$[FOC/x] : q = (1 + \frac{\gamma x}{k}) \tag{11}$$

$$[\texttt{recursive form}] : \; q = \beta E_{z'|z}[\pi'_k + \mathcal{AC}'_{k'} + q'(1-\delta)] \tag{12}$$

$$= \beta E_{z'|z} V_{k'}(z', k') \tag{13}$$

$$MC = MB \tag{14}$$

Optimality dictates that the firm set the marginal cost of investment (price and marginal adjustment cost) equal to discounted expected future marginal value, where such extra value comes from

- (i) $\pi'_{k'}$: being more **productive**,
- (ii) $\mathcal{AC}'_{k'}$: higher future capital r**educes future adjustment costs** so has a secondary benefit of avoiding costs later
- (iii) $q'(1-\delta)$: capital persists tomorrow, so we have the **scrap value**, valued at this internal shadow price (think of this a bit like an exchange rate converting units into profitability-terms not market prices).
- recursive structure means we can substitute for the summation version seen in Lecture 2 notes on Tobin model.

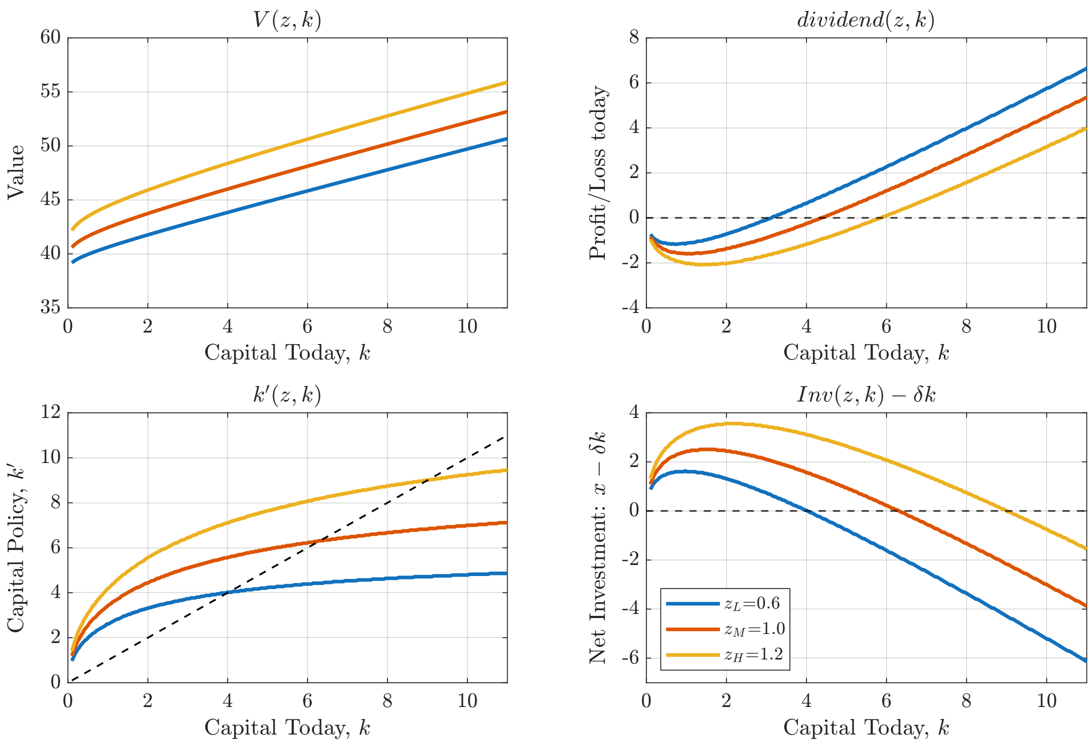
In general there is no need for average Q (V/K) - a ray from the origin to the value function, to coincide with marginal Q (the slope of the value function) unless under special circumstances. Visual inspection confirms curvature.

$$V(z,k) \neq constant \times k \Rightarrow \frac{\partial V(z',k')}{\partial k'} \neq \frac{V(z',k')}{k'}$$

1A.2 VFI solution

Comments : Some interesting shapes here given that our objective function is linear in profits (firms aren't risk averse), this comes from the quadratic adjustment cost function. Optimal investment is hump-shaped in current capital, suggesting for each level of z_t there is an optimum size the firm wants to grow to, and any excess capital will be sold to be returned as dividends. The net investment function should cross zero at the same level of capital for which the capital policy crosses 45 degrees: we have reached the steady state level of capital and will not adjust unless productivity fluctuates.

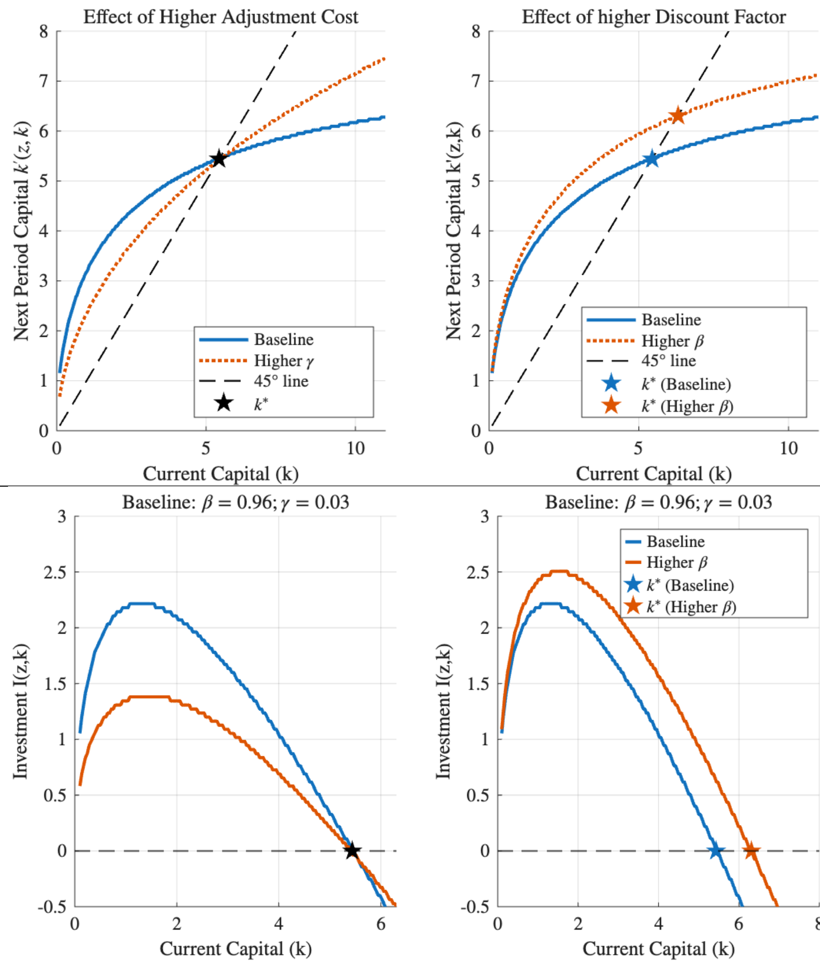
Figure 1: Baseline Tobin Model Solution



Notice something odd is happening in the profits chart at low levels of capital. If the firm is spending more than its revenues, who is paying for this? This is done with borrowing, but we haven't specified any financial structure. Silently we have been assuming there is a market for external finance all this time. Beware hidden assumptions! At the end of the section I plot what happens if we shut down negative dividends (aka borrowing). We can argue either way which is more realistic – firms often do borrow and make large losses in the growth phase, equally we could argue start-ups must finance themselves from retained earnings only since they have no credit record.

Optimal Size/45 degrees/zero note the curves of the Capital policy function should cut 45 degrees line at the optimal size at which the firm is content to remain. This should correspond to the points the net investment curves cut the x-axis

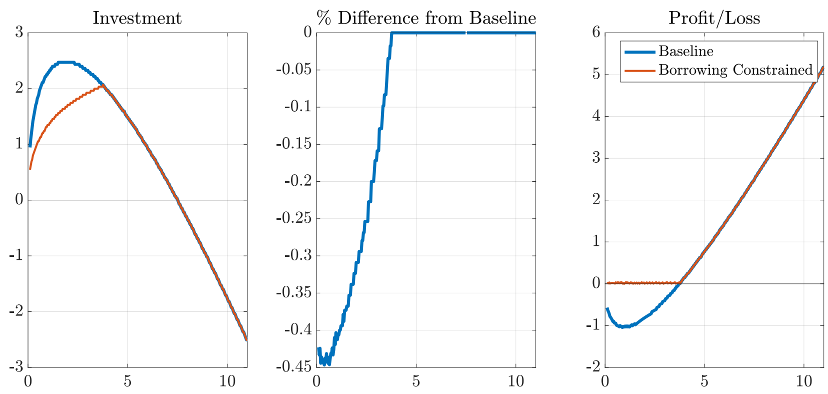
Figure 2: Capital Policy and Net Investment Response to Changing Parameters



Verbal arguements are fine, for the visual learners out there, consult the graphs above. If we adjust parameters of the Adjustment Cost / Tobin model we get:

- Let k^* denote the firm's desired optimal size given productivity. At this scale, the firm neither wants to grow or downsize, and replaces only depreciated capital, so $I = \delta k, netInv = I - \delta k = 0$
- **Higher γ surpresses investment/disinvestment** BUT firms converge to the same equilibrium sizes - growth is painful (as is contraction) so it happens more slowly in the high cost economy. For given expected profit stream, q, higher γ will lower the investment rate: $x/k = (1/\gamma)(q-1)$. Notice the orange capital policy line tries to stay close to the 45 degrees line as well, because changes are punished more by the cost function.
- **Higher patience β raises investment** through the standard Q equation: $(1+\frac{\gamma x}{k}) = \beta E_{z'|z} V_{k'}(z', k')$
- **Bonus**: Finance matters for growth! No-borrowing contraint: small firms invest less and grow slower.
- recall from class the so-called Stair Case plot of convergence to optimal size

Figure 3: Bonus: With and Without Borrowing Constraint at Zero



Crusoe

1B.1 recursive problem

Crusoe's problem to choose consumption and capital optimally can be written recursively as:

$$V(z,k) = \max_{c,k'} \left\{ u(c) + \beta \sum_{z'|z} P(z'|z) V(z',k'|z,k) \right\} \tag{15}$$

$$s.t. \quad c + \underbrace{[k' - (1-\delta)k]}_{I} = f(z,k) \tag{16}$$

$$(s.t. \quad c > 0) \tag{17}$$

NB: We can rewrite in terms of choice of k' only, absorbing the first constraint, and we know log utility will prohibit consumption of zero or below due to Inada conditions. However, for now, it is more helpful to keep the constraint explicit. The Lagrangian is then:

$$\boxed{\mathcal{L}(z,k,\lambda) = u(c) + \beta E_{z'|z} V(z',k') + \lambda\Big(f(z,k) - c - [k' - (1-\delta)k]\Big)} \tag{18}$$

First order conditions define optimality:

$$[FOC\ c] : \lambda = u_c \tag{19}$$

$$[k'] : \lambda = \beta E_{z'|z} \frac{\partial V(z',k')}{\partial k'} \tag{20}$$

Envelope Condtion [EC] needed, using **stationarity** [ST] between periods of value function:

$$[EC]: \frac{\partial V(z,k)}{\partial k} = \lambda(f_k + (1-\delta)) \tag{21}$$

$$[ST \Rightarrow]: \frac{\partial V(z',k')}{\partial k'} = \lambda'(f'_k + (1-\delta)) \tag{22}$$

Combine all parts to get the intertemporal optimality (new Euler equation):

$$\boxed{\begin{aligned} u_c &= \beta E_{z'|z}\left[MPK' + (1-\delta)\right]u'_c \\ \frac{1}{c} &= \beta E_{z'|z}\left[\alpha z' k'^{\alpha-1} + (1-\delta)\right]\frac{1}{c'} \end{aligned}} \qquad (23),\ (24)$$

Marginal Q imagine installing a small amount more of capital dk', this boosts NPV lifetime utility by $\beta E V_{k'}(z',k') \times dk'$ but we lose some utility from forgone consumption (via the budget $-dc = dk'$):

$$\overbrace{-(u_c \times dc)}^{\text{marginal loss}} + \overbrace{(\beta E_{z'|z} u'_c\left[f'_k + (1-\delta)\right] \times dk')}^{\text{marginal gain}} = \overbrace{0}^{\text{zero-arbitrage}} \tag{25}$$

$$\text{``}q\text{''} = \lambda = \beta E_{z'|z} V_{k'}(z',k') \tag{26}$$

Under the assumption of **utility maximisation** we will keep installing capital until the **marginal cost of foregone consumption** today u_c equals the **marginal value of extra future consumption** generated by extra capital installed. Recall the no arbitrage conditions we have talked about in class, this is exactly the same. **If Robinson is optimising, there should not be any feasible rearrangements** of capital which boosts Crusoe's utility of consumption over his lifetime.

This marginal value comes from f_k the **marginal product of extra capital** (we are slightly more productive) and the residual **scrap value**, $(1-\delta)dk$, all of which is converted to utility using u_c. Recall for small changes dx that (the slope at that point) $\times$ (the small change) will give the change in the function itself: $df = \frac{df}{dx}dx$.

Q still exists in a way in this model - the multiplier is still equal to the slope of the value function, however the **value function now inherits properties of the utility function** – Crusoe values dividends for their consumption value, not dividends for dividends sake.

Average and marginal Q will not coincide, except in very special circumstances. The shape of the value function should be a clue here: V is not linear in k:

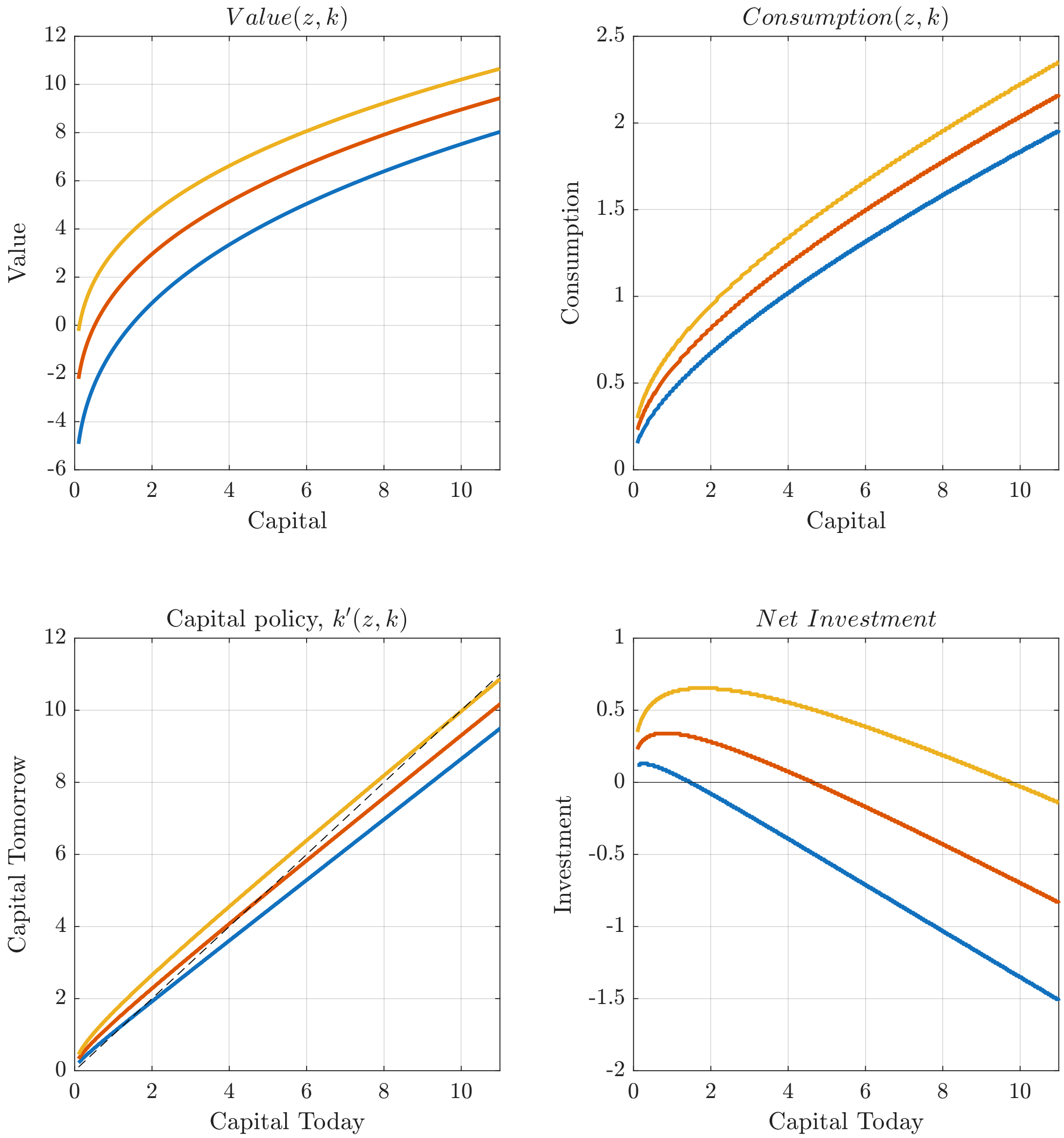
$$V(z,k) \neq constant \times k \Rightarrow \frac{\partial V(z',k)}{\partial k} \neq \frac{V(z',k)}{k}$$

and so Marginal Q is not equal to Average Q.

1B.2 VFI solution

The value function should inherit a **logarithmic slope** (envelope condition), and remember negative value is possible for low levels of capital, **Consumption is increasing and concave** in capital again natural given the consumption function inherits its shape from preferenes - Crusoe is risk averse, so the first units of consumption are very valuable to him, compared to higher levels of consumption. Capital policy is upward sloping a looks almost linear, however we see that **optimal investment is hump-shaped**. This is due to declining marginal returns to capital in the production function. For a given z, **CrusCorp has an optimal size**, beyond which Crusoe would prefer to consume more and do negative investment. The zeros of the $I(z,k) - \delta k$ net investment policy should match where $k'(z,k)$ cross the 45 degrees line: k^*

Figure 4: Value Function Iteration Solution to Crusoe's U-max Problem



Note here I am plotting **net investment**, that is, investment after accounting for depreciation

$$\mathcal{I}(z,k) - \delta k = (k' - k) = \Delta k_{t+1}$$

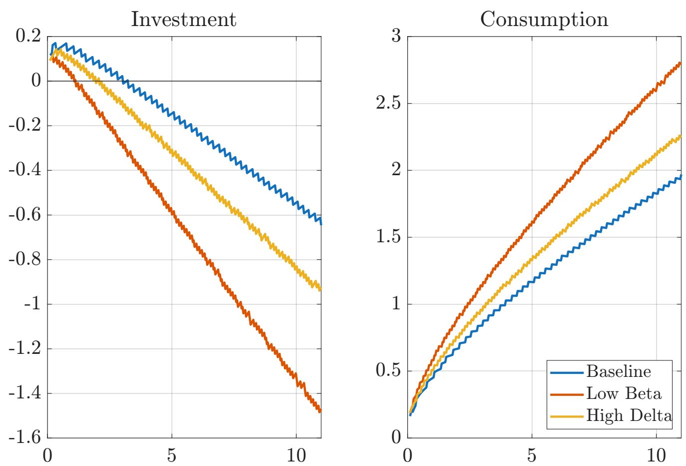
This is useful to see the firm's optimal size (for a fixed level of productivity z): When net investment is zero, the change in capital is zero, the firm has arrived after many steps adjusting k at optimal size. This is around $k = 2$ for the low productivity state, $k = 5$ for medium, and $k = 10.5$ for the most productive state.

$$\mathcal{I}(z, k^{opt-size}) = \delta k^{opt-size}$$

We can also calculate consumption at this optimal size (OS):

$$c(z, k^{OS}) + \delta k^{OS} = zf(k^{OS})$$

Figure 5: Investment and Consumption under alternative parameters



Here I plot investment and consumption only for one productivity state across the different cases. Recall the Euler equation:

$$u_c(c) = \beta E_{z'|z}\left[(\alpha z' k'^{\alpha-1} + (1-\delta))u_c(c')\right]$$

- Lower β will put less weight on investment (read: saving for future consumption) and more on consumption today in the intertemporal optimality condition, see Euler equation above: less investment, more consumption. If the RHS falls, to maintain equality, the $u_c(c)$ must fall, so c must rise (from concavity of u

- Higher δ reduces the net returns to saving (in capital), $[f_{k'} + (1-\delta)]$, shifting the the consumption-savings incentives towards consumption. So raises consumption and lowers investment, again by the same logic, to maintain equality, consumption today must adjust.

Newton

2.1 Sequential Problem and First Order Conditions

State Variables: Current capital stock: K_1, Current productivity state: θ_1

Choice Variables: Investment in period 1: I_1, Capital in period 2: K_2

Value of the Firm:

The value of the firm is the expected present discounted value of all dividends distributed to shareholders over the firm's lifetime.

Sequential Constrained Optimization Problem: The firm maximizes:

$$V(\theta_1, K_1) = \max_{I_1, K_2} \left\{ D_1 + \beta \mathbb{E}_{\theta_2|\theta_1} D_2 \right\}$$

subject to:

$$\begin{aligned}
D_1 &= \theta_1 K_1^\alpha - I_1 \\
D_2 &= \theta_2 K_2^\alpha + (1-\delta) K_2 \\
K_2 &= (1-\delta) K_1 + I_1
\end{aligned}$$

Substituting the constraints, we can remove one choice variable since it is no longer independent. I prefer to solve for K_2 instead of I_1, both are acceptable, but one is easier! The firm's problem becomes:

$$V(\theta_1, K_1) = \max_{K_2} \left\{ \theta_1 K_1^\alpha - (K_2 - (1-\delta)K_1) + \beta \mathbb{E}_{\theta_2|\theta_1} \left[\theta_2 K_2^\alpha + (1-\delta) K_2 \right] \right\}$$

First Order Conditions:

Taking the derivative with respect to K_2:

$$\begin{aligned}
1 &= \beta \mathbb{E}_{\theta_2|\theta_1} [\alpha \theta_2 K_2^{-(1-\alpha)} + (1-\delta)] \\
MC &= MB
\end{aligned}$$

Interpretation: The first order condition states that the marginal cost of investing one unit today (which is 1) must equal the expected discounted marginal benefit. The marginal benefit consists of two components: operating profit boost, and scrap value. The firm invests up to the point where the cost of foregoing one unit of dividends today equals the expected discounted return from that investment tomorrow.

2.2 Numerical Solution Using Newton's Method

Given parameters: $K_1 = 5$, $\beta = 0.96$, $\alpha = 0.33$, $\delta = 0.025$, $\theta \in \{0.7, 1.0, 1.3\}$

The firm starts in the low state with $\theta_1 = 0.7$.

Newton's Method Setup:

We need to solve the system of equations given by the first order conditions. For period 1, starting from state θ_L. Call the solution to the function, X:

$$f(X) = -1 + \beta \sum_j P(\theta_j|\theta_L) \left[\alpha \theta_j X^{\alpha-1} + (1-\delta)\right] = 0$$

Newton's method iteration:

$$X^{(n+1)} = X^{(n)} - \frac{f(X^{(n)})}{f'(X^{(n)})}$$

The derivative is:

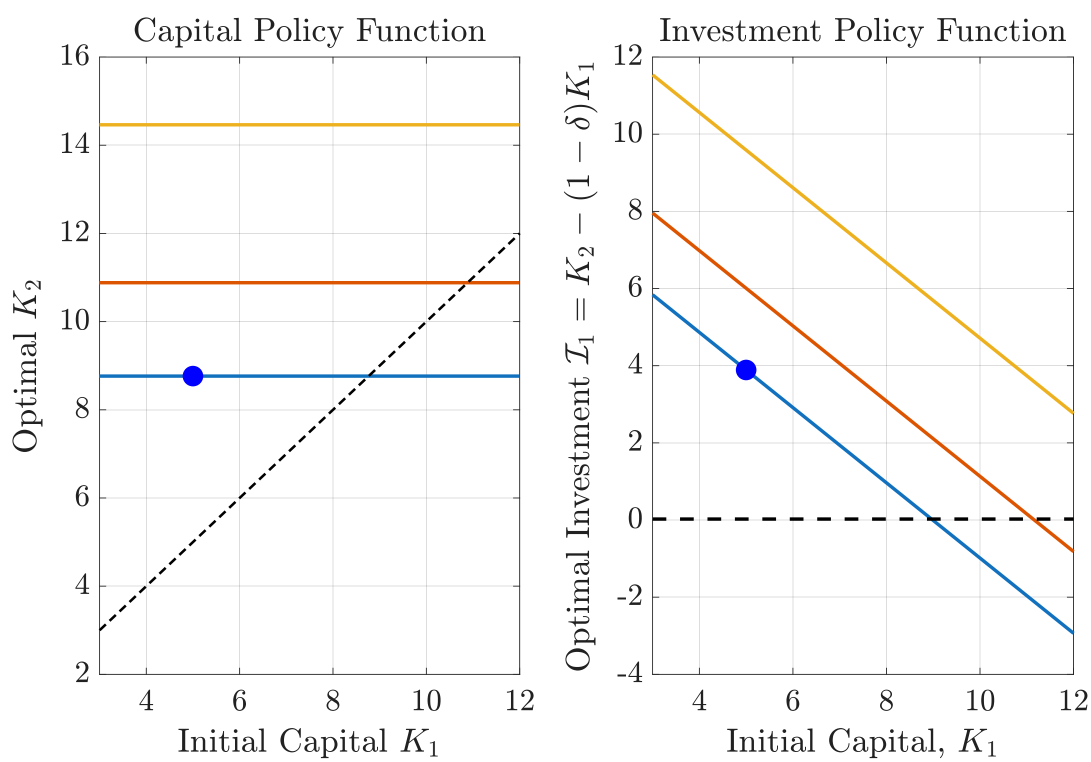
$$f'(X) = \beta \sum_j P(\theta_j|\theta_L) \left[\alpha(\alpha-1)\theta_j X^{\alpha-2}\right]$$

Note the SOC of the firm problem, or the slope of our newton function $f'(X)$ above should be negative at a maximum!

Computational Results:

Using the given transition probabilities from state θ_L: $P(\theta_L|\theta_L) = 0.60$, $P(\theta_M|\theta_L) = 0.25$, $P(\theta_H|\theta_L) = 0.15$.

Figure 6: Newton Method Solution for K_2; all 3 productivities shown



Interpretation:

The policy function $I_1(K_1, \theta_L)$ shows how optimal investment varies with the initial capital stock when the firm starts in the low productivity state. Key features to observe:

- For $K_1 < K_2^*$: As K_1 increases, the gap between optimal K_2 and initial K_1 falls, so the need to invest also falls in K_1. Starting above this level and it is more profitable to sell capital for dividends than suffer diminishing marginal productivity further.

- NB that K_2 is just **an expression of fixed model parameters**, so does not vary with K_1, hence a flat line.

- Optimality condition is **user cost** in disguise!

$$1 = \beta[E_{\theta_2|\theta_1}[MPK(K_2)] + (1-\delta)] \tag{27}$$

$$\frac{1}{\beta} - (1-\delta) = E[MPK(K_2)] \tag{28}$$

$$\Rightarrow (1+r) - (1-\delta) = E[MPK(K_2)] \tag{29}$$

$$(r+\delta) = \textit{expected } MPK(K_2) \tag{30}$$

$$UC = EMPK_2 \tag{31}$$

$$\tag{32}$$

Algebraic result:

$$1 = \beta\mathbb{E}_{\theta_2|\theta_1}[\alpha\theta_2 K_2^{-(1-\alpha)} + (1-\delta)] \Rightarrow K_2^* = \left[\frac{\alpha\beta E(\theta_2|\theta_1 = \theta_L)}{1-\beta(1-\delta)}\right]^{1/(1-\alpha)} \tag{33}$$

$$E(\theta_2|\theta_1 = \theta_L) = 0.6(0.7) + 0.25(1.0) + 0.15(1.3) = 0.865 \tag{34}$$

$$K_2^* = 8.76. \tag{35}$$

To show the power of Newton's method, in a few iterations the solver has found the optimum within a tiny tolerance of 1e-10:

Newton K2 (high precision): 8.76432286083796263654

Theory K2 (high precision): 8.76432286083796796561

For a broad class of models which can be expressed as some (possibly complicated) function $f(\mathbf{X}) = 0$, we only need $f(\mathbf{X})$ to be continuous (no jumps), existence of the derivative which is not too close to zero, and a good guess not too far from the true root.