Mathematical Toolkit

Investment, Finance and Asset Prices ECON5068

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Essentials: Calculus of small changes

Partial Derivatives: $\frac{\partial F(x, y)}{\partial x}$ - derivative with respect to x, treating y (all other inputs) as constants.

- Slope of F in direction x (see figure next slide)

Notation: Let $\partial F / \partial x = F_x$

Total Differential: $dF(x_1, .., x_n) = \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 ... + \frac{\partial F}{\partial x_n} dx_n$

- Ex: How do profits change (dF) for small changes in labour, capital, materials, and energy, when all move at once (dL, dK, dM, dE)?

See **Practice Questions** for examples (and eventually solutions)

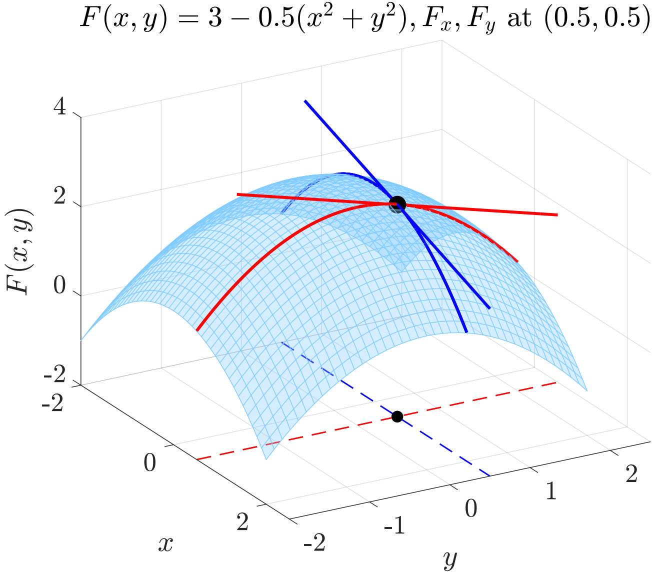


Figure 1: Partial Derivatives

Partial and Total Derivatives, a worked example

When differentiating **with respect to x, treat y as constant (and vice versa)**

$$
\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\partial}{\partial x}(3 - 0.5x^2 - 0.5y^2) \\
&= 0 - 0.5(2)x - 0 \qquad (1) \\
&= -x \\
\frac{\partial F}{\partial y} &= -y \qquad (2)
\end{aligned}
$$

Total change dF for small changes in all inputs (at a point):

$$
\begin{aligned}
dF(x, y) &= (-x)dx + (-y)dy \\
dF(0.5, 0.5) &= (-0.5)dx + (-0.5)dy
\end{aligned}
$$

Optimisation and First-Order Conditions

Consider the function

$$F(x, y) = 3 - 0.5x^2 - 0.5y^2.$$

First-order conditions:

$$F_x = \frac{\partial F}{\partial x} = -x = 0,$$
$$F_y = \frac{\partial F}{\partial y} = -y = 0.$$

Solution: The stationary point is at

$$(x^*, y^*) = (0, 0),$$

with

$$F(0, 0) = 3.$$

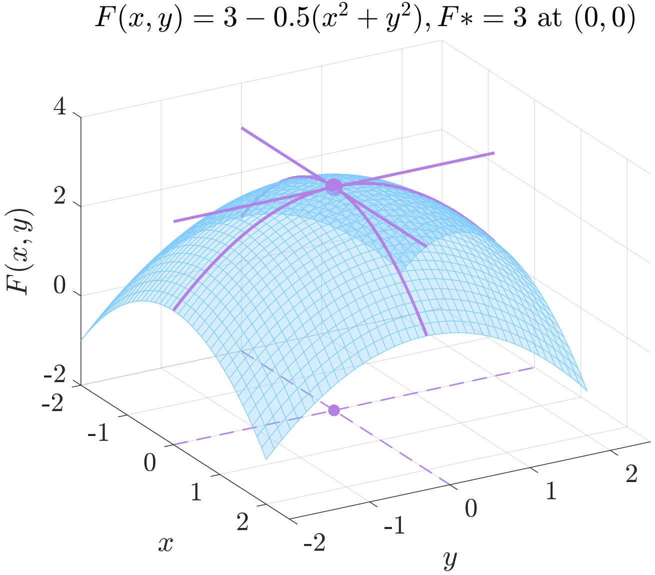


Figure 2: Maximum achieved! partial derivatives (slopes) equal zero

Essentials: Optimisation Rules

First Order Condition (FOC): For maximum/minimum, $f'(x) = 0$

Second Order Condition: $f''(x) < 0$ for maximum, $f''(x) > 0$ for minimum

Constrained Optimization: Lagrangian $\mathcal{L} = f(x, y) - \lambda[g(x, y) - c]$

FOCs for Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \frac{\partial \mathcal{L}}{\partial y} = 0 \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \tag{3}$$

Shadow Price: $\lambda^* = \dfrac{\partial f^*}{\partial c}$ - value of relaxing constraint

Essentials: Derivatives and Rules

Chain Rule: If $z = f(g(x))$, then $\dfrac{dz}{dx} = f'(g(x)) \cdot g'(x)$

Product Rule: $(uv)' = u'v + uv'$

Quotient Rule: $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

Power Rule: $\dfrac{d}{dx}x^n = nx^{n-1}$

Exponential: $\dfrac{d}{dx}e^x = e^x$, $\dfrac{d}{dx}e^{ax} = ae^{ax}$

Logarithm: $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$

Essentials: Discounting and Present Value

Discount Factor: $\beta = \dfrac{1}{1+r}$ where r is interest rate

Present Value: $PV = \dfrac{FV}{(1+r)^t}$ where FV is future value in period t

Net Present Value: $NPV = \sum_{t=0}^{T} \dfrac{CF_t}{(1+r)^t}$ where CF_t is cash flow

Infinite Horizon: $PV = \sum_{t=0}^{\infty} \beta^t (X_t)$ with $\beta < 1$

Geometric Series: $\sum_{t=0}^{\infty} ar^t = \dfrac{a}{1-r}$ if $|r| < 1$

Discount Rate from Factor: $r = \dfrac{1-\beta}{\beta}$

Essentials: Concavity and Convexity

Concave Function: $f''(x) < 0$ - shaped like C

- Diminishing marginal returns
- Any local maximum is global maximum

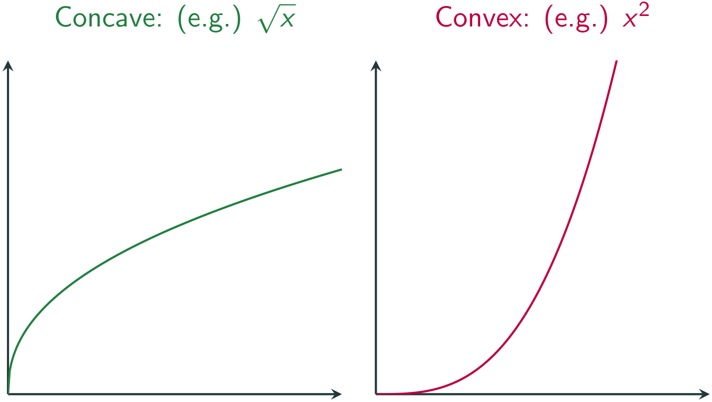
Convex Function: $f''(x) > 0$ - shaped like V

- Increasing marginal cost
- Any local minimum is global minimum

Concave $\Rightarrow$ **Decreasing Marginal Product**

Convex $\Rightarrow$ **Increasing Marginal Cost**

Reminder: Concave vs. Convex shapes



- Tip: con**C**ave looks a bit like a C, con**V**ex looks like a V?

Essentials: Envelope Theorem

Setup: Optimization problem with parameter α

$$\max_x f(x, \alpha) \text{ gives solution } x^*(\alpha)$$

Value Function: $V(\alpha) = F(x^*(\alpha), \alpha)$

Envelope Theorem:

$$\frac{dV}{d\alpha} = \frac{\partial F}{\partial \alpha}|_{x=x^*(\alpha)}$$

Key Insight: Ignore indirect effects, they cancel due to FOC

Constrained Version:

$$\frac{dV}{d\alpha} = \frac{\partial \mathcal{L}}{\partial \alpha}|_{\text{optimum}}$$

Why It Works: At optimum, $\frac{\partial \mathcal{L}}{\partial x} = 0$, so indirect effects vanish

Essentials: Dynamic Programming Basics

Bellman Equation: $V(s) = \max_a\{u(s,a) + \beta EV[s'(s,a)]\}$

Value Function: $V(s_t)$ - maximum value starting from state s_t

Policy Function: $a^*(s)$ - optimal action in state s

State Variables: Predetermined at start of period (e.g., K_t)

Control/Choice Variables: Chosen each period (e.g., I_t)

Euler Equation: Links optimal choices across time periods

Operator: maps one function to another: $V(x) \to \mathcal{T}[V(x)]$ e.g. differentiation $\mathcal{D}(3x^2) \to 6x$.

Fixed Point: a function V^* such that $V^* = \mathcal{T}[V^*]$.
(Value function is the fixed point of the Bellman operator.)

Parameters, Exogeneous and Endogeneous Variables

Parameters: Fixed numbers within the model which do not change and the decision maker cannot control. eg. depreciation δ. Think of them as the rules of the game.

Exogeneous Variable: `From the Greek exo (outside)`. A moving variable determined outside the model and taken as given when solving. Usually follow a stochastic process. For example, productivity, market demand. Think: environmental states.

Endogeneous Variable: `From the Greek endon (inside)`. A variable determined inside the model, as a result of choices and/or equilibrium. e.g. consumption, hours of work, k_{t+1}. Think: choices.

Predetermined Varaibles and Stochastic Processes

Predetermined Variable: Endogenous variables whose current value was decided in the past, part of the state today and constrain current choices. e.g. the debt you owe at the start of the period or k_t.

Stochastic Processes: Random variables whose future realizations are uncertain. We usually specify the distribution and motion of the process, realised value happen with some controlled probability, expectations can be formed, usually written as $E(Z_{t+1}|Z_t)$

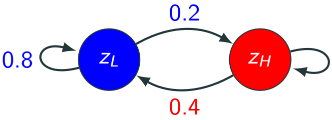
Transition Matrix. Let $z \in \mathbf{Z}_{[n\times 1]}$, $Pr(Z_{t+1}|Z_t) = \mathbf{P}_{n\times n}$

Transition Matrix: 2 State Example

Discrete economic processes: can only move between **states**

E.g: income employed/jobless, productivity shocks: $\mathbf{Z} = [z_L = 0.75, z_H = 1.00]$

Transition between states is **probabilistic**:



$0.6 = Pr(Z_{t+1} = z_H | Z_t = z_H)$

As a **matrix of conditional probabilities**:

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \leftarrow z_t \tag{4}$$

$$z_{t+1} \uparrow \tag{5}$$

Let X_t be the distribution (%) across states, e.g. [0.30, 0.70]

$$X_{t+1} = X_t \mathbf{P} = [0.52, 0.48] \tag{6}$$

Also Useful: Expectations and Time

Conditional Expectation: $E_t[X_{t+1}]$ - expectation of X_{t+1} given info at time t

Law of Iterated Expectations: $E_t[E_{t+1}[X]] = E_t[X]$

Time Subscripts: $E_0, E_1, E_2, \ldots$ where subscript indicates information set

Updating Rule: $E_{t+1}[X] = E_t[X] +$ new information

Also Useful: Identities and Shortcuts

Log Approximation: $\ln(1+x) \approx x$ for small x

Growth Rates: $\frac{d\ln(x)}{dt} = \frac{1}{x}\frac{dx}{dt} =$ growth rate

Elasticity: $\epsilon = \frac{\partial \ln y}{\partial \ln x} = \frac{x}{y}\frac{\partial y}{\partial x}$

Quadratic Formula: $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Taylor Approximation:

$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$

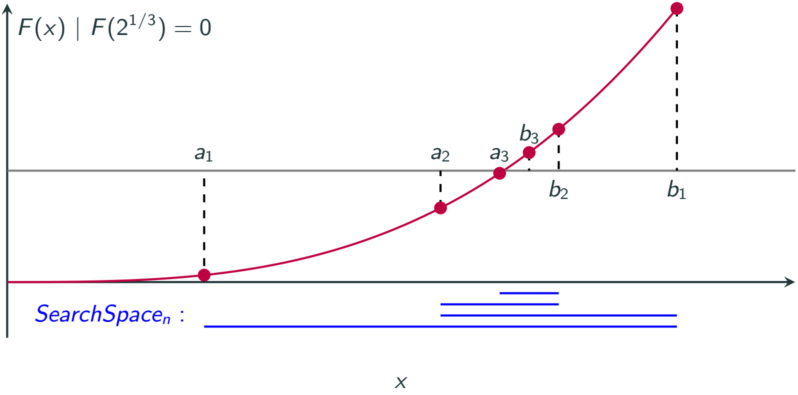
L'Hôpital's Rule: If $\frac{f(x)}{g(x)} \to \frac{0}{0}$, then $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$

Also Useful: Finding Roots of Functions

Two numerical methods to find the **zeros (roots)** of some complicated function

- **Bisection**: deletes half the search space with each iteration!
- **Newton-Raphson**: find the root of a linear approximation (linear = easy!)
- only requirement: we can evaluate $f(x)$ for some guess x_{guess}

Also Useful, Rootfinder 1: Bisection / Binary Search



Rule: new guess x_{n+1} : avg $a_n + b_n$, split the range, update end points

$$S_1 = [a_1 = 0.5, b_1 = 1.7] \quad \Rightarrow x_1 = 0.5(0.5 + 1.7) \quad f(1.1) = -ive \quad (7)$$

$$S_2 = [1.1, 1.7] \quad \Rightarrow x_2 = 0.5(1.1 + 1.7) \quad f(1.4) = +ive \quad (8)$$

$$S_3 = [1.1, 1.4] \quad \Rightarrow x_3 = 0.5(1.1 + 1.4) \quad f(1.25) = -ive \quad (9)$$

Deletes half the search space each iteration!!

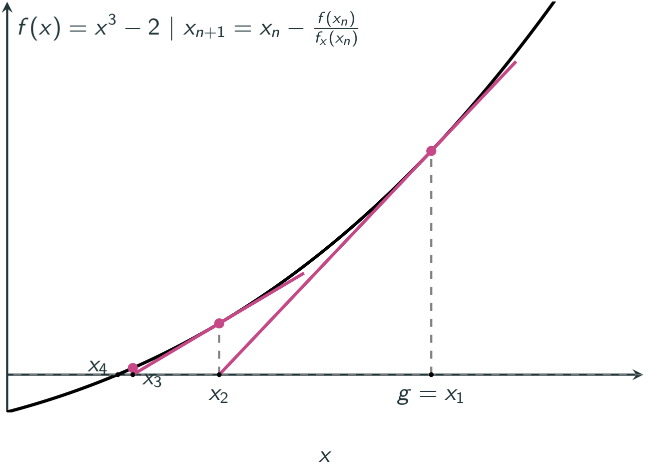
Maybe not so useful in the wild...

"After [bike is stolen, police do nothing] I found a chatroom thread among Cambridge computer scientists, one of whom had also been told that unless he could pin down the moment of theft no one would look at the footage. He said he had tried to explain sorting algorithms to police — he was a computer scientist, after all.

You don't watch the whole thing, he said. You use a binary search. You fast forward to halfway, see if the bike is there and, if it is, zoom to three quarters of the way through. But if it wasn't there at the halfway mark, you rewind to a quarter of the way through.

It's very quick. In fact, he had pointed out, if the CCTV footage stretched back to the dawn of humanity it would probably have only taken an hour to find the moment of theft. **This argument didn't go down well**." - Times Article 2024

Rootfinder 2: Newton-Raphson / Newton's Method



Newton's method uses linear approximation (at guess g):
$f(x)|_{x=g} \approx f(g) + f'(g)(x - g)$. Solves this much easier approximation for x, which becomes the new guess, g.