Tutorial 2: Q-model and Lagrangian Practice

Investment, Finance, Asset Prices ECON5068

Suggested Solutions

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Overview

1. Cobb-Douglas Production Function
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   - Characteristics
2. Question 1.a: Unconstrained Optimization
   - Choice of labour
   - Optimal labour
3. Question 1.b: Constrained Optimization
   - Defining the dividends
   - Dynamic optimization problem
   - The Lagrangean method
   - Optimal Investment Rule

Cobb-Douglas Production Function

$$Y_t = A_t K_t^{\alpha} L_t^{\beta} \tag{1}$$

where K_t; L_t denote capital and labour respectively (the inputs), A_t the level of productivity, and Y_t the output.

- A_t can be seen as the technology level (for the same amount of capital and labour, a higher level of technology A_t will result in a higher output). In our case, we consider that the technology level is given, that is, even though it changes between different periods (t), we cannot choose its amount.
- α and β denote the capital and labour shares of output, respectively. The higher the share, the higher the importance of the respective input in the production of output. We consider that α, $\beta > 0$ and $\alpha + \beta \leq 1$.

Cobb-Douglas Production Function

Decreasing Marginal Productivity: The higher the amount of the input, the lower the increase in output due to an increase of the input. Recall that the marginal productivity is given by the partial derivative of the output in terms of the input:

$$MPK_t = \frac{\partial Y_t}{\partial K_t} = \alpha A_t K_t^{\alpha-1} L_t^{\beta} \quad MPL_t = \frac{\partial Y_t}{\partial L_t} = \beta A_t K_t^{\alpha} L_t^{\beta-1} \tag{2}$$

- The previous analysis is for the effects of the change in 1 of the inputs on the output. What about a change in both/all inputs?
- Either have constant returns to scale/function is homogeneous of degree 1 (if $\alpha + \beta = 1$) or decreasing returns to scale/function is homogeneous of degree < 1 (if $\alpha + \beta < 1$). In mathematical terms:

$$F(aK, aL) = aF(K, L) \quad \text{or} \quad F(aK, aL) < aF(K, L), \forall a > 0 \tag{3}$$

Marginal Rate of technical substitution (MRTS): It shows the rate at which we can substitute one of the inputs by the other, while maintaining the output constant. This can be achieved through the ratio of marginal productivities:

$$MRTS(K, L) = -\frac{\Delta L}{\Delta K} = \frac{MPK}{MPL} = \frac{\alpha L}{\beta K} \tag{4}$$

- In the previous condition, it shows by how much we can decrease L, if we increase K by one unit, in order to keep Y constant. Notice that the $MRTS(K, L)$ will be different from the $MRTS(L, K)$

Static Profits: Choice of Labour

$$\pi_t = A_t K_t^{\alpha} L_t^{\beta} - w_t L_t \tag{5}$$

- **Goal**: Choose the amount of labour L in order to maximize the profits in each period t (consider the price of output is normalized to 1).
- **Step 1**: This is an optimization problem (in this case, a maximization problem), which we solve by calculating the FOC with respect to labour, set it equal to 0 and solve for L:

$$\frac{\partial \pi_t}{\partial L_t} = 0 \Rightarrow MPL_t - w_t = 0 \Rightarrow \beta A_t K_t^{\alpha} L_t^{\beta-1} = w_t \Rightarrow A_t K_t^{\alpha} L_t^{\beta-1} = \frac{w_t}{\beta} \tag{6}$$

Static Profits: Optimal labour

Step 2: Use the previous condition to express profits in terms of labour and wages:

$$\pi_t = A_t K_t^{\alpha} L_t^{\beta} - w_t L_t \Rightarrow \pi_t = L_t(A_t K_t^{\alpha} L_t^{\beta-1} - w_t) \Rightarrow \pi_t = L_t \left(\frac{w_t}{\beta} - w_t \right) \quad (7)$$

Step 3: Go back to condition (6) in step 1 and express L as a function of the remaining variables:

$$A_t K_t^{\alpha} L_t^{\beta-1} = \frac{w_t}{\beta} \Rightarrow L_t^{\beta-1} = \frac{w_t}{\beta A_t K_t^{\alpha}} \Rightarrow \left(L_t^{\beta-1} \right)^{-1} = \left(\frac{w_t}{\beta A_t K_t^{\alpha}} \right)^{-1} \Rightarrow$$

$$(L_t)^{1-\beta} = \frac{\beta A_t K_t^{\alpha}}{w_t} \Rightarrow L_t^* = \left(\frac{\beta A_t}{w_t} \right)^{\frac{1}{1-\beta}} K_t^{\frac{\alpha}{1-\beta}} \quad (8)$$

Static Profits: Optimal Profits

Step 4: Replace Optimal labour of step 3 in the profit condition of step 2:

$$\pi_t = \left(\frac{\beta A_t}{w_t}\right)^{\frac{1}{1-\beta}} K_t^{\frac{\alpha}{1-\beta}} \left(\frac{w_t}{\beta} - w_t\right) \Rightarrow \pi_t = Z_t K_t^{\frac{\alpha}{1-\beta}} \quad (8)$$

where $Z_t = \left(\frac{\beta A_t}{w_t}\right)^{\frac{1}{1-\beta}} \left(\frac{w_t}{\beta} - w_t\right)$

- Notice that we do not need to follow step 2, we can go directly from step 1 to step 3, and then replace in the profit condition (5). **Try it out!**

Constrained Optimization

After the choice for the labour, we now need to choose the amount of capital in order to maximize profits. However, unlike labour, the choice of capital amount is not completely free. We need to invest in order to accumulate capital, and investment is costly. Even if we want to simply maintain the current amount of capital, we still need to invest, due to the depreciation of capital over time.

Step 1: Define the variable to maximize. In the exercize, we are informed that the goal is to maximize the dividends. Since we need to pay workers, make investments, and cover the costs of the investments, the dividends will be the remaining component of the profits after making these payments:

$$Div_t = \pi_t - I_t - AC_t = Z_t K_t^{\frac{\alpha}{1-\beta}} - I_t - \frac{\phi}{2} \frac{I_t^2}{K_t} \quad (9)$$

Dynamic optimization problem

Step 2: Define the dynamic problem. The goal is not to maximize the dividends in period t, but instead, to maximize the sum of the discounted value of the dividends, subjected to our constraints:

$$\max E_0 \sum_{t=0}^{\infty} \left(\frac{1}{R}\right)^t \left(Z_t K_t^{\frac{\alpha}{1-\beta}} - I_t - \frac{\phi}{2}\frac{I_t^2}{K_t}\right) \tag{10}$$

s.t.

$$K_{t+1} = I_t + (1-\delta)K_t \tag{11}$$

- The choice variables (which we can define to maximize the dividends) are the capital amount for next period (K_{t+1}), and the investment in the current period (I_t). K_t is a state variable (either is given, or results from the choices in the previous periods)

Step 3: Choose solving method. There are 2 ways to solve the previous problem: 1 - The Lagrangean function; 2 - The Value function. In this exercize, we are asked to use the Lagrangean function method, which involves incorporating our constraint (11) into the objective function (10):

$$\mathcal{L} = \max_{\{K_{t+1}, I_t\}} E_0 \sum_{t=0}^{\infty} \left(\frac{1}{R}\right)^t \left(Z_t K_t^{\frac{\alpha}{1-\beta}} - I_t - \frac{\phi}{2}\frac{I_t^2}{K_t} - \lambda_t(K_{t+1} - I_t - (1-\delta)K_t)\right) \tag{12}$$

Where λ_t is the Lagrange multiplier.

First Order Conditions

Step 4: To optimize, we need to calculate the F.O.C. in terms of our choice variables and the multiplier, and set them equal to 0:

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow \frac{-K_{t+1} + I_t + (1-\delta)K_t}{R^t} = 0 \Rightarrow K_{t+1} = I_t + (1-\delta)K_t \qquad (13)$$

$$\frac{\partial \mathcal{L}}{\partial I_t} = 0 \Rightarrow = \frac{1}{R^t}\left(\frac{-2\phi}{2}\frac{I_t}{K_t} - 1 + \lambda_t\right) = 0 \Rightarrow \lambda_t = 1 + \frac{\phi I_t}{K_t} \qquad (14)$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Rightarrow \frac{-\lambda_t}{R^t} + \frac{1}{R^{t+1}}\left(\frac{\alpha}{1-\beta} Z_{t+1} K_{t+1}^{\frac{\alpha+\beta-1}{1-\beta}} + \frac{\phi}{2}\frac{I_{t+1}^2}{K_{t+1}^2} + \lambda_{t+1}(1-\delta)\right) = 0 \qquad (15)$$

Don't Forget: In this last condition, you need to take into account the K_{t+1} in period t, and the K_t in period $t+1$

Derive the Optimal Investment Rule

Step 5: Use the FOC to get the optimal investment rule. Let's start by trying to simplify condition (15), which can be rewritten as:

$$\lambda_t = \frac{1}{R}\left(\pi_K(K_{t+1},) + \frac{\phi}{2}\frac{I_{t+1}^2}{K_{t+1}^2} + \lambda_{t+1}(1-\delta)\right) \tag{16}$$

where $\pi_K(K_{t+1},) = \frac{\alpha}{1-\beta} Z_{t+1} K_{t+1}^{\frac{\alpha+\beta-1}{1-\beta}}$. If we substitute condition (14) into condition (16), we get the **optimal investment rule**:

$$1 + \frac{\phi I_t}{K_t} = \frac{1}{R}\left(\pi_K(K_{t+1},) + \frac{\phi}{2}\frac{I_{t+1}^2}{K_{t+1}^2} + \lambda_{t+1}(1-\delta)\right) \tag{17}$$

- The left-hand side gives the marginal cost, while the right-hand side gives the expected discounted marginal benefit of investment

Comparison with the exercize from the lectures

Notice that our solution is very similar to the one you get from the exercize in Lecture 2 (see slide 16). Taking into account that the $\beta = \frac{1}{R}$, and that $q = \lambda$, the difference results from 2 points: 1 - on the left-hand side, the marginal adjustment cost; 2 - on the right-hand side, the expected discounted marginal benefit includes the additional reduction in the cost in terms of investment.

Both these differences result from the capital adjustment costs function (AC): In our case, the adjustment cost also depends on the level of capital. Everything else is similar to the exercize of Lecture 2.