1 Question 1: Application to Expected Utility and Uncertainty

1.1 Expected Utility

In the exercise, we are informed that the agent's utility is dependent on the amount of money, x, and is given by $u(x) = \sqrt{x}$.

The agent's money is given by his initial wealth of 9$, and the lottery ticket that pays 16$ with probability $\frac{1}{4}$ and will pay 0$ with probability $\frac{3}{4}$.

In the first question, we are asked to calculate the expected utility, which is given by:

$$E[u(x)] = \frac{1}{4} \cdot \sqrt{25} + \frac{3}{4} \cdot \sqrt{9} = \frac{5}{4} + \frac{9}{4} = 3.5 \tag{1}$$

Notice that this is not the same as the utility of the expected value:

$$u(E[x]) = \sqrt{\frac{1}{4} \cdot 25 + \frac{3}{4} \cdot 9} = \sqrt{13} \approx 3.6 \tag{2}$$

1.2 Indifference Under Uncertainty

In the second question, we are asked what is the lowest price at which the agent would sell the lottery ticket. On one hand, selling the lottery ticket would give the agent additional money without uncertainty. But on the other hand, he would also lose the possibility of winning the lottery.
The lowest price at which the agent is willing to sell the lottery ticket is the one that makes him indifferent between selling or owning the lottery ticket:

$$u(9 + p) = E[u(x)] \Rightarrow \sqrt{9 + p} = 3.5 \Rightarrow p = 3.25 \tag{3}$$

1.3 Absolute Risk Aversion

For the third question, we are asked to calculate if the agent has a constant, increasing, or decreasing absolute risk aversion, given by the Arrow-Pratt measure.
We start by calculating the absolute risk-aversion, using the formula that was given:

$$r_A = -\frac{U''(x)}{U'(x)} = \frac{\frac{1}{4}x^{-\frac{3}{2}}}{\frac{1}{2}x^{-\frac{1}{2}}} = \frac{1}{2}x^{-1} \tag{4}$$

To understand how the absolute risk aversion changes with x, we can simply calculate the derivative:

$$\frac{dr_A}{dx} = -\frac{1}{2}x^{-2} = -\frac{1}{2x^2} < 0 \tag{5}$$

The agent has a decreasing absolute risk aversion: the higher the amount of money x, the lower the absolute risk aversion.
Notice that these results will depend on the utility function:

- risk-averse: concave utility $\iff u(E[x]) \geq E[u(x)] \iff \frac{dr_A}{dx} < 0$
- risk-neutral: linear utility $\iff u(E[x]) = E[u(x)] \iff \frac{dr_A}{dx} = 0$
- risk-lover: convex utility $\iff u(E[x]) \leq E[u(x)] \iff \frac{dr_A}{dx} > 0$

You can confirm this with $u(x) = x$ or $u(x) = x^2$. Try it out.

2 Question 2: Application to Investment under Uncertainty

2.1 Investment Options

The investment scenario consists of an initial cost of $-20M\$$ at time $t = 0$, followed by uncertain outcomes at $t = 1$:

- With probability $\frac{1}{3}$: receive $1M\$$ at $t = 1$, then $0.5M\$$ per period forever
- With probability $\frac{1}{3}$: receive $2M\$$ at $t = 1$, then $2M\$$ per period forever
- With probability $\frac{1}{3}$: receive $3M\$$ at $t = 1$, then $5M\$$ per period forever

2.2 Overall Scenario

Step 1: Establish the investment scenario and respective cash flows (see previous slide).
Step 2: Define the probabilities for the different scenarios, in the different periods. For year $t = 2$, we have 3 possible scenarios: High, Moderate, and Low. They all have equal probability, which implies each has $p = \frac{1}{3}$. After the event in the second year, the cash flows for all future periods will be the same for future years, and will only differ according to the scenario in year $t = 2$.
Step 3: Establish the calculation for the value of the project. We should use the Net Present Value (NPV).

2.3 NPV: Net Present Value

NPV is the discounted sum of expected cash flows of the investment project:

$$NPV = \sum_{t=0}^{\infty} \frac{R_t}{(1+r)^t} \tag{6}$$

where R_t is the returns net costs in period t and r is the discount rate.

Notice that this is very similar to the dynamic problems we solved in previous tutorials (can think of $\frac{1}{(1+r)^t}$ as β^t, and R_t as π_t or $u(c_t)$).

Difference: The outcomes in each period do not depend on our choices, but the probabilities of the different scenarios. Our only decision will be whether to invest or not invest in the project.

Step 4: Establish the NPV for the specific project:

$$NPV = -20 + \frac{1}{3}[CF_L + CF_M + CF_H] \tag{7}$$

Step 5: Calculate the respective cash flows for each scenario, using the formula from condition (6):

$$CF = \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \frac{R_3}{(1+r)^3} + \dots \tag{8}$$

2.4 Perpetuity Condition

$$CF_L = \frac{1}{1.1} + \frac{0.5}{(1.1)^2} + \frac{0.5}{(1.1)^3} + \dots = \frac{1}{1.1} + \frac{0.5}{(1.1)^2}\left[1 + \frac{1}{1.1} + \frac{1}{(1.1)^2} + \dots\right] \tag{9}$$

Step 6: Use the perpetuity condition to simplify the calculations. Recall that, after the second year, the amount received in each period is the same, and is received forever. We can refer to this amount as a perpetuity. The general formula for the present value of a perpetuity is given by:

$$\frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots = \frac{C}{r} \tag{10}$$

Notice that the condition (10) is very similar to what we have in the [] brackets in condition (9).

2.5 Perpetuity vs Convergence of Geometric Series

When we use the perpetuity condition (10), we can simplify the cash flows of condition (9):

$$CF_L = \frac{1}{1.1} + \frac{0.5}{(1.1)^2}\left[1 + \frac{1}{0.1}\right] = \frac{1}{1.1} + \frac{0.5}{(1.1)^2}[11] = 5.45 \tag{11}$$

As an alternative to the perpetuity condition, we can use the condition of convergence of geometric series. When $|b| < 1$, we have that:

$$C + C \cdot b + C \cdot b^2 + C \cdot b^3 + \dots = \frac{C}{1-b} \tag{12}$$

Notice that if you define $C = 1$ and $b = \frac{1}{(1+r)}$, and since $\left|\frac{1}{(1+r)}\right| < 1$, we can use condition (12) to simplify the terms in [] brackets in condition (9):

$$CF_L = \frac{1}{1.1} + \frac{0.5}{(1.1)^2}\left[\frac{1}{1 - \frac{1}{1.1}}\right] = \frac{1}{1.1} + \frac{0.5}{(1.1)^2}[11] = 5.45 \tag{13}$$

We should achieve the same result using either the perpetuity condition, or the convergence of geometric series condition.

Step 7: Repeat the calculation of the cash flows for each scenario:

$$\begin{aligned} CF_M &= \frac{2}{1.1} + \frac{2}{(1.1)^2} + \ldots = \frac{2}{1.1} + \frac{2}{(1.1)^2}\left[1 + \frac{1}{1.1} + \frac{1}{(1.1)^2} + \ldots\right] \\ &= \frac{2}{1.1} + \frac{2}{(1.1)^2}\left[1 + \frac{1}{0.1}\right] = \frac{2}{1.1} + \frac{2}{(1.1)^2}[11] = 20 \end{aligned} \tag{14}$$

$$\begin{aligned} CF_H &= \frac{3}{1.1} + \frac{5}{(1.1)^2} + \ldots = \frac{3}{1.1} + \frac{5}{(1.1)^2}\left[1 + \frac{1}{1.1} + \frac{1}{(1.1)^2} + \ldots\right] \\ &= \frac{3}{1.1} + \frac{5}{(1.1)^2}\left[1 + \frac{1}{0.1}\right] = \frac{3}{1.1} + \frac{5}{(1.1)^2}[11] = 48.18 \end{aligned} \tag{15}$$

Step 8: Replace the calculated cash flows in condition (7) to get the NPV, and make a decision whether to invest or not invest:

$$NPV = -20 + \frac{1}{3}[5.45 + 20 + 48.18] = -20 + \frac{1}{3}[73.63] = 4.54 \tag{16}$$

Decision Rule:

- If $NPV > 0 \Rightarrow$ Accept the project
- If $NPV < 0 \Rightarrow$ Reject the project
- If $NPV = 0 \Rightarrow$ Indifferent between accepting or rejecting the project

Since the NPV of the project is > 0, we accept the project.

2.6 Change in the Initial Cost/Investment

We are informed that the initial investment/cost of $20M\$$ was underestimated by 20%. Therefore, the actual cost has to be such that:

$$C \cdot (1 - 0.2) = -20 \Rightarrow C = -\frac{20}{0.8} = -25 \tag{17}$$

Careful: Notice that this is not the same as saying that the actual cost is 20% higher than initially estimated. If that was the case, we would have:

$$C = -20 \cdot (1 + 0.2) = -24 \neq -25 \tag{18}$$

We need to recalculate the NPV with the new cost:

$$NPV = -25 + \frac{1}{3}[5.45 + 20 + 48.18] = -25 + \frac{1}{3}[73.63] = -0.46 \tag{19}$$

Since the $NPV < 0$, we reject the project.

2.7 Change in the Cash Flows

In the last question, we are informed that the cash flows after the second year in the high scenario were overestimated. As a result we need to start by recalculating the cash flows under this scenario:

$$\begin{aligned} CF_H &= \frac{3}{1.1} + \frac{3.5}{(1.1)^2} + \ldots = \frac{3}{1.1} + \frac{3.5}{(1.1)^2}\left[1 + \frac{1}{1.1} + \frac{1}{(1.1)^2} + \ldots\right] \\ &= \frac{3}{1.1} + \frac{3.5}{(1.1)^2}\left[1 + \frac{1}{0.1}\right] = \frac{3}{1.1} + \frac{3.5}{(1.1)^2}[11] = 34.55 \end{aligned} \tag{20}$$

Then, recalculate the NPV:

$$NPV = -20 + \frac{1}{3}[5.45 + 20 + 34.55] = -20 + 20 = 0 \tag{21}$$

Since the $NPV = 0$, we are indifferent between accepting or rejecting the project.