Practice Questions - Suggested Solutions

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Question 1

Consumer maximizes utility $u(x, y) = xy$ subject to budget constraint $x + 4y = 16$.
Setting up the Lagrangean:

$$\mathcal{L} = xy + \lambda(16 - x - 4y) \tag{1}$$

First order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = y - \lambda = 0 \Rightarrow y = \lambda \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - 4\lambda = 0 \Rightarrow x = 4\lambda \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 16 - x - 4y = 0 \tag{4}$$

From the first two conditions: $x = 4y$
Substituting into the constraint:

$$4y + 4y = 16 \Rightarrow 8y = 16 \Rightarrow y^* = 2 \tag{5}$$

Therefore: $x^* = 4(2) = 8$

$$\boxed{\text{The optimal consumption bundle is } (x^*, y^*) = (8, 2), \lambda^* = 2, F^* = 2 \cdot 8 = 16.} \tag{6}$$

Question 2

Firm maximizes profit with production function $f(K, L) = K^a L^b$, output price p, wage w, rental rate r, subject to constraint $K = L$.
Profit function:

$$\pi = pK^a L^b - wL - rK \tag{7}$$

With constraint $K = L$:

$$\pi = pK^{a+b} - wK - rK = pK^{a+b} - (w + r)K \tag{8}$$

First order condition:

$$\frac{d\pi}{dK} = p(a+b)K^{a+b-1} - (w+r) = 0 \tag{9}$$

Solving for optimal K:

$$\boxed{K^* = \left(\frac{p(a+b)}{w+r}\right)^{\frac{1}{1-(a+b)}}} \tag{10}$$

$$\boxed{\pi(K^*, N(K*)) = p\left(\frac{p(a+b)}{w+r}\right)^{\frac{a+b}{1-(a+b)}} - (w+r)\left(\frac{p(a+b)}{w+r}\right)^{\frac{1}{1-(a+b)}}} \tag{11}$$

Therefore: $L^* = K^*$

Question 3

Silicon Valley firm produces microchips with cost function $c(y)$. Fraction $(1-\alpha)$ are defective, so only αy chips can be sold at price p.
Profit function:

$$\pi = p \cdot \alpha y - c(y) \tag{12}$$

First order condition:

$$\frac{d\pi}{dy} = p\alpha - c_y(y) = 0 \tag{13}$$

This gives optimal output y^* where $c'(y^*) = p\alpha \Rightarrow y^*(\alpha) = c_y^{-1}(p\alpha)$.
To find effect of increase in quality (increase in α):

$$\boxed{\frac{d\pi^*}{d\alpha} = \underbrace{\cancel{\frac{\partial\pi}{\partial y}\frac{dy^*}{d\alpha}}}_{EC:=0} + \frac{\partial\pi}{\partial\alpha} = py^*} \tag{14}$$

An increase in production quality increases profit by py^*.

Question 4

Firm maximizes $f(x,y) = xy$ subject to $x^2 + ay^2 = 1$.
Lagrangean:

$$\mathcal{L} = xy + \lambda(1 - x^2 - ay^2) \tag{15}$$

First order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda x = 0 \Rightarrow y = 2\lambda x \tag{16}$$
$$\frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda ay = 0 \Rightarrow x = 2\lambda ay \tag{17}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - x^2 - ay^2 = 0 \tag{18}$$

From first two conditions: $\frac{y}{x} = \frac{2\lambda x}{2\lambda ay} = \frac{x}{ay}$
This gives: $y^2 = \frac{x^2}{a}$, so $y = \frac{x}{\sqrt{a}}$
Substituting into constraint:

$$x^2 + a \cdot \frac{x^2}{a} = 1 \Rightarrow 2x^2 = 1 \Rightarrow x^* = \frac{1}{\sqrt{2}} \tag{19}$$

Therefore: $y^* = \frac{1}{\sqrt{2a}}$
Maximum profit: $f^* = x^*y^* = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2a}} = \frac{1}{2\sqrt{a}}$

$$\boxed{(x^*, y^*, \pi^*) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2a}}, \frac{1}{2\sqrt{a}}\right)} \tag{20}$$

Change in profits: brute force When $a = 1$: $f^*(1) = \frac{1}{2}$ When $a = 1.1$: $f^*(1.1) = \frac{1}{2\sqrt{1.1}} \approx \frac{1}{2.095} \approx 0.477$
Brute force: Extra profit from change: $\boxed{f^*(1.1) - f^*(1) = 0.477 - 0.5 = -0.023}$

Change in profits: Envelope Theorem

$$EnvelopeTheorem : \frac{\partial V(a)}{\partial a} = \frac{\partial \mathcal{L}}{\partial a} = -\lambda^*(y^*)^2 \tag{21}$$
$$NB : \lambda^* \textit{comes from FOC[x or y]} = 1/2 \tag{22}$$
$$= -(1/2)(1/2) \tag{23}$$
$$= -(1/4) \tag{24}$$
$$\therefore \boxed{dV_{ET} \approx \frac{\partial V}{\partial a} da = -(1/4) * (0.1) = -0.025} \tag{25}$$

NB: this second part is subtly different to relaxing the constraint $\neq g(x, y) = c + dc$, instead we are reshaping the feasible set, in this case making it more cyclindrical, which the objective punishes. I invite you to graph the functions at `https://www.desmos.com/3d`

Question 5

Firm maximizes $f(x, y) = xy$ subject to $2x + y = 3$.

Lagrangean:

$$\mathcal{L} = xy + \lambda(3 - 2x - y) \tag{26}$$

First order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda = 0 \Rightarrow y = 2\lambda \tag{27}$$
$$\frac{\partial \mathcal{L}}{\partial y} = x - \lambda = 0 \Rightarrow x = \lambda \tag{28}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 3 - 2x - y = 0 \tag{29}$$

From first two conditions: $y = 2x$
Substituting into constraint:

$$2x + 2x = 3 \Rightarrow 4x = 3 \Rightarrow x^* = \frac{3}{4} \tag{30}$$

Therefore: $y^* = \frac{3}{2}$
Maximum profit: $f^* = x^* y^* = \frac{3}{4} \cdot \frac{3}{2} = \frac{9}{8}$
The multiplier is $\lambda^* = x^* = \frac{3}{4}$

$$\boxed{\text{The change in optimal profit from one more unit of input is } \lambda^* = \frac{3}{4}.} \tag{31}$$

Question 6

Factory produces $Q(x, y) = 50x^{1/2}y^{1/2}$ units with budget constraint $x + y = 80$.
Lagrangean:

$$\mathcal{L} = 50x^{1/2}y^{1/2} + \lambda(80 - x - y) \tag{32}$$

First order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = 25x^{-1/2}y^{1/2} - \lambda = 0 \tag{33}$$
$$\frac{\partial \mathcal{L}}{\partial y} = 25x^{1/2}y^{-1/2} - \lambda = 0 \tag{34}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 80 - x - y = 0 \tag{35}$$

From first two conditions:

$$25x^{-1/2}y^{1/2} = 25x^{1/2}y^{-1/2} \Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow y = x \tag{36}$$

Substituting into constraint:

$$x + x = 80 \Rightarrow x^* = y^* = 40 \tag{37}$$

Maximum output: $Q^* = 50(40)^{1/2}(40)^{1/2} = 50 \cdot 40 = 2000$
The multiplier is $\lambda^* = 25(40)^{-1/2}(40)^{1/2} = 25$

$$\boxed{\text{If allocation decreased by \$1000: Change in maximum output} \approx -\lambda^* \cdot 1 = -25 units.} \tag{38}$$

Question 7

Static Profits are

$$\pi(A, K, L) = \max_{K,L} \left\{ AK^{1/2}L^{1/2} - wL - rK \right\} \tag{39}$$

The necessary condition for max. is given by the first order conditions w.r.t K & L:

$$\frac{\partial \pi}{\partial K} = 0 :\Rightarrow (1/2)AK^{-1/2}L^{1/2} = r \tag{40}$$
$$\frac{\partial \pi}{\partial L} = 0 :\Rightarrow (1/2)AK^{1/2}L^{-1/2} = w \tag{41}$$

Taking the ratio of the above two eq's we see that

$$\frac{L}{K} = \frac{r}{w} \tag{1}$$

Since the wage-rental ratio is 1 every period,

$$K = L \tag{42}$$

Now we can write profit as a function of only A and K. A is a constant in this model so we can consider it as a parameter. The static profits are then:

$$\pi(K) = AK - wK - rK = (A - w - r)K = \tilde{A}K \tag{43}$$

where $\tilde{A} = A - w - r$. Adjustment cost is convex in the adjustment rate, and scale with firm-size:

$$\mathcal{AC}(I, K) = \frac{\phi}{2}\left(\frac{I}{K}\right)^2 \cdot K \tag{44}$$

The Lagrangean function is

$$\mathcal{L}(K_t, I_t, \lambda_t) = \mathbb{E}_0 \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \left[\tilde{A}_t K_t - pI_t - \frac{\gamma}{2}\frac{I_t^2}{K_t} - \lambda_t\left(K_{t+1} - (1-\delta)K_t - I_t\right)\right] \tag{L}$$

Note that you don't really need the expectations operator here since all the variables are deterministic but it's a good practice to always have $E[.]$ since the absence of any randomness is the exception rather than the rule. Moreover $E[X] = X$ if X is not a random variable.
The necessary condition for optimality is that the gradients for the objective and the constraint functions are a constant λ multiple of each other. These are given by the first order conditions w.r.t the variables K_{t+1} and I_t. Remember K_{t+1} will enter also where we see K_t next period i.e. "$(K_t)_{+1}$"

$$\frac{\partial}{\partial I_t} = 0 :\Rightarrow \texttt{[MCMB]}: \; p + \gamma\frac{I_t}{K_t} = \lambda_t \tag{45}$$

$$\frac{\partial}{\partial K_{t+1}} = 0 :\Rightarrow \texttt{[TQ]}: \; \lambda_t = \frac{1}{1+r}\mathbb{E}_t\left[\tilde{A}_{t+1} + \frac{\gamma}{2}\frac{I_{t+1}^2}{K_{t+1}^2} + \lambda_{t+1}(1-\delta)\right] \tag{46}$$

$$q_t = \beta\mathbb{E}_t \sum_{k=0}^{\infty}\beta^k(1-\delta)^k\left[\tilde{A}_{t+k+1} + \frac{\gamma}{2}\frac{I_{t+k+1}^2}{K_{t+k+1}^2}\right] \tag{47}$$

Marginal Q or Tobin's Q is just the multiplier or the RHS of the above equation. See Lecture Notes for interpretation. Some extra capital has two parts to its value

- (1) expected NPV of stream of future MPKs ($\tilde{A}_{t+k+1}$)
- *plus* (2) marginal adjustment costs are falling in K, so installing today reduces marginal adjustment costs in the future, $\frac{\partial}{\partial K}\frac{\partial \mathcal{AC}(I,K)}{\partial I} < 0$.

In the labs, we solve firm's problem by constructing a functional equation (Bellman equation) and then iterating on this using value function iteration. Alternatively, given initial state K_0 and the values of parameters, we can solve the above two Lagrangean first order conditions along with the constraint:

$$\frac{\partial}{\partial \lambda_t} = 0 :\Rightarrow K_{t+1} = (1-\delta)K_t + I_t \tag{48}$$

using Euler equation approaches like extended path, perturbation, projection etc sometimes coupled with a nonlinear equation solver like Newton's method. These methods work well when the FOC's are sufficient (concave obj. & convex constr. set) which is not always the case. In the later part of this course, we will look at firm models with non convex adjustment costs where you have no choice but to resort to value function iteration.