Practice Questions - Optimization

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1 The Multiplier and the Shadow Price

When solving constrained optimization problems, we also derive values for the multiplier. What information does this multiplier represent? **The multipliers measure the sensitivity of the optimal value of the objective function to changes in the right hand sides of the constraints.** Consider a simple problem with two variables and one constraint,

$$\max f(x, y) \tag{1}$$

$$g(x, y) = a \tag{2}$$

Here a, the right hand side of the constraint, is some parameter that can vary from problem to problem. Let $x^*(a), y^*(a)$ be the solution to this optimization problem and $\lambda^*(a)$ be the multiplier. So the optimal value of the objective function is $f\Big(x^*(a), y^*(a)\Big)$, then the multiplier $\lambda^*(a)$ measures the rate of change of the optimal value of f with respect to the parameter a,

$$\lambda^*(a) = \frac{d}{da} f\Big(x^*(a), y^*(a)\Big) \tag{1}$$

If f is a profit function and the constraint is on inputs, then the multiplier represents the change in the optimal profit resulting from the availability of one more unit of input. The multiplier tells us how valuable one more unit of input would be to firm's profits. Alternatively, it tells us the maximum amount the firm would be willing to pay to acquire another unit of input. For this reason, it is also called the shadow price of input. This shadow price may infact be more important to the firm than the actual external market price of the input.

If you have more than one constraint, then the multiplier is defined in terms of partial derivatives. For a two constraint problem this is

$$\lambda_i^*(a_1, a_2) = \frac{\partial}{\partial a_i} f\Big(x^*(a_1, a_2), y^*(a_1, a_2)\Big) \tag{2}$$

where $i = 1, 2$ and a_1, a_2 are right hand side parameters for the two constraints.

Proof of equation (1)

Consider the constrained optimization problem

$$\max f(x,y) \quad s.t: \quad g(x,y) = a \tag{3}$$

This is a parameterized constraint, where the parameter is a. Let $x^*(a), y^*(a)$ be the solution to this optimization problem and $\lambda^*(a)$ be the multiplier. So the optimal value of the objective function is denoted by $f(x^*(a), y^*(a))$.The usual FOCs of the Lagrangean are:

$$[\mathcal{L}_x]: \ \frac{\partial f}{\partial x}(x^*(a)) = \lambda^*(a)\frac{\partial g}{\partial x}(x^*(a)) \tag{3}$$

$$[\mathcal{L}_y]: \ \frac{\partial f}{\partial y}(y^*(a)) = \lambda^*(a)\frac{\partial g}{\partial y}(y^*(a)) \tag{4}$$

Taking derivative of the constraint wrt a, at the optimum, $g(x^*(a), y^*(a)) = a$, we have

$$\frac{d}{da}(a) = \frac{d}{da}g(x^*(a), y^*(a)) \tag{4}$$

$$1 = \frac{\partial g}{\partial x}\frac{dx^*}{da} + \frac{\partial g}{\partial y}\frac{dy^*}{da} \tag{5}$$

Where we have applied the chain rule. Applying the chain rule again to $f(x^*(a), y^*(a))$,

$$\frac{d}{da}f(x^*(a), y^*(a)) = \frac{\partial f}{\partial x}\frac{dx^*}{da} + \frac{\partial f}{\partial y}\frac{dy^*}{da} \tag{5}$$

$$= \left[\lambda^*(a)\frac{\partial g}{\partial x}(x^*(a))\right]\frac{dx^*}{da} + \left[\lambda^*(a)\frac{\partial g}{\partial y}(y^*(a))\right]\frac{dy^*}{da} \tag{6}$$

$$= \lambda^*(a)\left[\frac{\partial g}{\partial x}\frac{dx^*}{da} + \frac{\partial g}{\partial y}\frac{dy^*}{da}\right] \tag{7}$$

$$= \lambda^*(a) \cdot [1] \tag{8}$$

$$= \lambda^*(a) \tag{9}$$

where we have used results from eq's (3) and (4) in the second line and eq. (5) in the fourth line. The lagrange multiplier is thus the change in the maximal value of $f(.)$ w.r.t the parameter a:

$$\lambda^*(a) = \frac{d}{da}f(x^*(a), y^*(a)) \tag{10}$$

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2 Envelope Theorem

Envelope theorem is a theorem which describes how the *optimal value of the objective function* in a parameterized optimization problem *changes as one of the parameters changes.* Eq. (1) that describes the lagrange multiplier is a particular case of this Envelope theorem. For an unconstrained optimization problem,

$$\max f(x; a) \tag{11}$$

the Envelope Theorem says that at the optimum, we can ignore indirect effects:

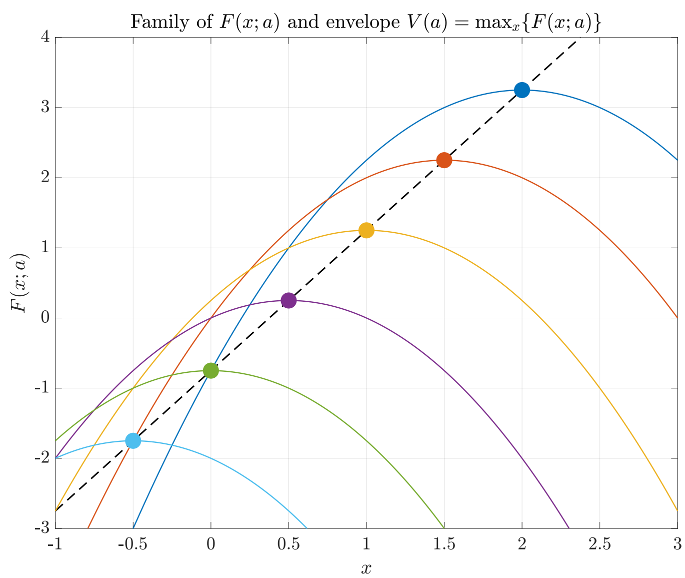
$$\frac{d}{da} f(x^*(a)) = \frac{\partial}{\partial a} f(x^*(a)) \tag{6}$$

As you can see, the lagrange multiplier defined in eq. (1) looks quite similar to this theorem. However, the **partial derivative** on the RHS of (6) is much easier to deal with than the **total derivative** on the LHS of (1).

To see this explicitly: define the **Value Function** as the optimised objective given parameters:

$$V(a) = \max_x f(x; a) = f(x^*(a); a)$$

Figure 1: $F(x; a) = (a - x)^2 + x + a \mid a = -1.0, -0.5, ..., 1.5$



We want to find how the value function moves with changes in the parameter a, $V'(a) = \frac{d}{da} f(x^*(a), a)$.

$$\frac{d}{da} f(x^*(a), a) = \frac{\partial f}{\partial x}\frac{dx^*}{da} + \frac{\partial f}{\partial a} \tag{12}$$

$$= (\textit{indirect effect on optimal choice}) + (\textit{direct effect of } a \textit{ on } f) \tag{13}$$

$$= 0 + \frac{\partial f}{\partial a} \tag{14}$$

The first term here captures the indirect effect of a change in a on f through changes in optimal choice x.

Due to optimality (the definintion of the value function itself) **this term is zero:**

$$\frac{\partial}{\partial x} f(x^*(a); a) = 0 \tag{15}$$

This is only true when we are evaluating at x^*

Example 1.

What will be the effect of a unit increase in the parameter a on the maximum value of

$$f(x; a) = -x^2 + 2ax + 4a^2 \tag{16}$$

First, let's compute the answer directly without using the Envelope Theorem. For optimal f, we need to take the F.O.C w.r.t x:

$$f'(x) = -2x + 2a = 0 \Rightarrow x^* = a \tag{17}$$

The maximum value is then

$$\begin{align} f(x^*(a)) &= -a^2 + 2a^2 + 4a^2 \tag{18} \\ &= 5a^2 \tag{19} \end{align}$$

To see what the effect of an increase in a is on this optimal value, we can take the derivative w.r.t a

$$\frac{d}{da} f(x^*(a)) = 10a \tag{20}$$

So the answer is that optimal f increases at the rate of $10a$ with increase in a.
By applying the Envelope Theorem, we could bypass all these steps by using eq. (6) as

$$\begin{align} \frac{d}{da} f(x^*(a)) &= \frac{\partial f(x^*(a))}{\partial a} \tag{21} \\ &= \frac{\partial}{\partial a} \left[-x^{*2} + 2ax^* + 4a^2\right] \tag{22} \\ &= 2x^* + 8a \tag{23} \\ &= 2a + 8a \quad [\text{since } x^* = a] \tag{24} \\ &= 10a \tag{25} \end{align}$$

In most cases, Envelope Theorem deals with constrained optimization problems. The general Envelope Theorem for a constrained optimization problem says that

$$\frac{d}{da} f(x^*(a)) = \frac{\partial}{\partial a} \mathcal{L}(x^*(a), \lambda^*(a); a) \tag{7}$$

where L is the Lagrangean function for the problem evaluated at the optimal values $x^*(a)$ and $\lambda^*(a)$. Note that you still need to find optimal values (x^* and λ^*) and the derivative is w.r.t to the parameter a and not the variables. Again we can ignore the indirect effects in chain-rule-terms, and the total derivative becomes the partial only.

$$\frac{d}{da}f(x^*(a)) = \frac{d}{da}\mathcal{L}(x^*(a), \lambda^*(a); a) \tag{26}$$

$$= \cancel{\frac{\partial \mathcal{L}}{\partial x}}\frac{dx}{da} + \cancel{\frac{\partial \mathcal{L}}{\partial \lambda}}\frac{d\lambda}{da} + \frac{\partial \mathcal{L}}{\partial a} \tag{27}$$

$$= 0 + 0 + \frac{\partial \mathcal{L}}{\partial a} \tag{28}$$

See Practice Questions 3,4,5 and 6 to understand the application of Envelope Theorem in constrained optimization problems.

3 Practice Questions

A set of optimization questions for practice. Try to solve on your own before looking at the solution.

Q. 1

Consider a consumer who wishes to maximize her utility given by $U(x, y) = xy$ subject to the budget constraint $x + 4y = 16$. Find the optimal consumption bundle for this consumer (x^*, y^*).

Q. 2

Consider a firm that uses two inputs, capital K and labor L, to produce output using the production function $f(K, L) = K^a L^b$. The output price is p and the input prices are w for labor and r for capital. The firm is constrained by a law that says that it must use exactly the same number of units of both inputs. Find the optimal input combination that maximizes firm's profits.

Q. 3

A Silicon Valley firm produces an output of microchips denoted by y and has a cost function $c(y)$. Assume that the first and second derivatives of the cost function are strictly positive. Of the chips it produces, a fraction $1 - \alpha$ are unavoidably defective and cannot be sold. Working chips can be sold at a price p and the microchip market is highly competitive. How will an increase in production quality affect the firm's profit?

Q. 4

Consider a firm that maximizes profit function, $f(x, y) = xy$, subject to the input constraint, $x^2 + ay^2 = 1$, where x, y are two inputs in the production. Find the extra profits garnered by the firm when the constant a changes from 1 to 1.1 if the input combination is chosen optimally. Assume that optimal inputs and the multiplier are strictly positive.

Q. 5

A firm maximizes its profit function, $f(x,y) = xy$, subject to the input constraint set, $2x + y = 3$. Find the maximum value of profit and the change in the optimal profit resulting from one more unit of input.

Q. 6

If x thousand dollars is spent on labour and y thousand dollars is spent on equipment a factory produces $Q(x,y) = 50x^{1/2}y^{1/2}$ units of output. How should $\$80,000$ be allocated between labour and equipment to yield the largest possible output? Estimate the change in maximum output if the allocation decreased by \$1000.

Q. 7

Consider the following firm level investment problem. The firm lives forever and has a discount factor given by $1/R < 1$. The price of capital is one. The production function is given by $AK^{1/2}L^{1/2}$, where A denotes the productivity level, K is capital and L labour. The firm rents capital stock at rate r and hires labor at the wage rate w. Assume that the wage-rental ratio is 1 every period.
The firm faces capital adjustment costs in the form of $\frac{\gamma}{2}\frac{I^2}{K}$, where I denotes investment and $\gamma > 0$ is a constant parameter. The firm faces a constant depreciation rate for capital equal to δ and investment creates new capital with a one period lag.

- Derive profits as a function of only the productivity level A and capital K. (HINT: write the static problem to choose K and N within the period first)
- Define the lagrangean function and derive the first order conditions.