
Lab 3 Solutions: Value Function Iteration on a Grid
Investment Finance and Asset Prices ECON 5068

Grid-Search Value Function Iteration Solution

The Bellman equation characterizing the firm's problem can be written as

$$V(K) = \max_{K',I} \left[K^\alpha - I - \frac{\phi}{2} I^2 + \beta V(K') \right] \tag{1}$$

where $I = K' - (1-\delta)K$. We are denoting next period values using the prime notation. Time subscripts are redundant since this is a stationary model producing stationary policies. Why? At any point of time, whether it is $t = 1$ or $t = 1000$, the problem is the same and is summarized by just the state K and hence time has no meaning.

Rewriting the value function in terms of capital terms only,

$$V(K) = \max_{K'} \left[K^\alpha - (K' - (1-\delta)K) - \frac{\phi}{2} \left(K' - (1-\delta)K\right)^2 + \beta V(K') \right] \tag{2}$$

We need to find optimal policy functions for next period capital and investment, both functions of the state variable K:

$$K'(K) \text{ and } I(K)$$

We would also like to understand how the value function $V(K)$ and other policy functions change with changes in the parameters of the model (comparative statics).

This problem is identical to the cake eating problem and can be solved by iterating on the functional equation (2). The procedure known as **Value Function Iteration** can be summarized in the following steps:

1. Create a grid for capital, $\mathcal{K} = \{K_1, \dots K_m\}$.
2. Guess that $V_0(K) = 0$ for all $K \in \mathcal{K}$. In other words, initialize value function to be zeros.

3. Update the value function and obtain $V_1(K)$. Specifically, for each of the $1, 2, \ldots m$ state space points in $\mathcal{K}$

   (a) Compute the right hand side term in eq. (2) to be maximized:

   $$RHS = \left[K^\alpha - (K' - (1-\delta)K) - \frac{\phi}{2}\left(K' - (1-\delta)K\right)^2 + \beta V_0(K')\right]$$

   This is evaluated by restricting K' to also fall in the grid $\mathcal{K}$. These values are all conditional on a given K.

   (b) Updated value function is then the maximum of this RHS.

   $$\mathtt{V}_1(K) = \max\{RHS\} = \max_{\mathtt{K}' \in \mathcal{K}} \left[K^\alpha - (K' - (1-\delta)K) - \frac{\phi}{2}\left(K' - (1-\delta)K\right)^2 + \beta \mathtt{V}_0(K')\right]$$

   We can find the maximum using Brute force grid search.[1] We will also save the points on the capital grid where the maximum is obtained to get next period capital $K'(K)$.

4. Compare the intial guess, V_0, and the update V_1 by computing the distance (error) between them.

   $$\mathtt{error} = \max_K |V_0 - V_1|$$

5. If the `error` is smaller than some tolerance level (a very small number, usually `1e-8`),

   $$\mathtt{error} \leq \mathtt{tol} = \mathtt{1e\text{-}8}$$

   then the algorithm has converged. The optimal value function is given by V_1.

6. If the error is greater than the tolerance level,

   $$\mathtt{error} > \mathtt{tol} = \mathtt{1e\text{-}8}$$

[1] To implement this grid search, we will employ Matlab's inbuilt function max that finds both the maximum value and also the location of this maximum.

then the error is not small enough. We will update our guess as:

$$\mathtt{V}_0 = \mathtt{V}_1$$

and proceed to next iteration (step 3).

This is the discretized value function iteration algorithm.[2] Note that we are restricting K' to take value in the grid $\mathcal{K}$. A finer grid will of course lead to a better approximation.

The algorithm is implemented in the Matlab code called investmentcode.m. Run it to get the policy functions, you can then play with the parameters of the model to see how policy functions change.

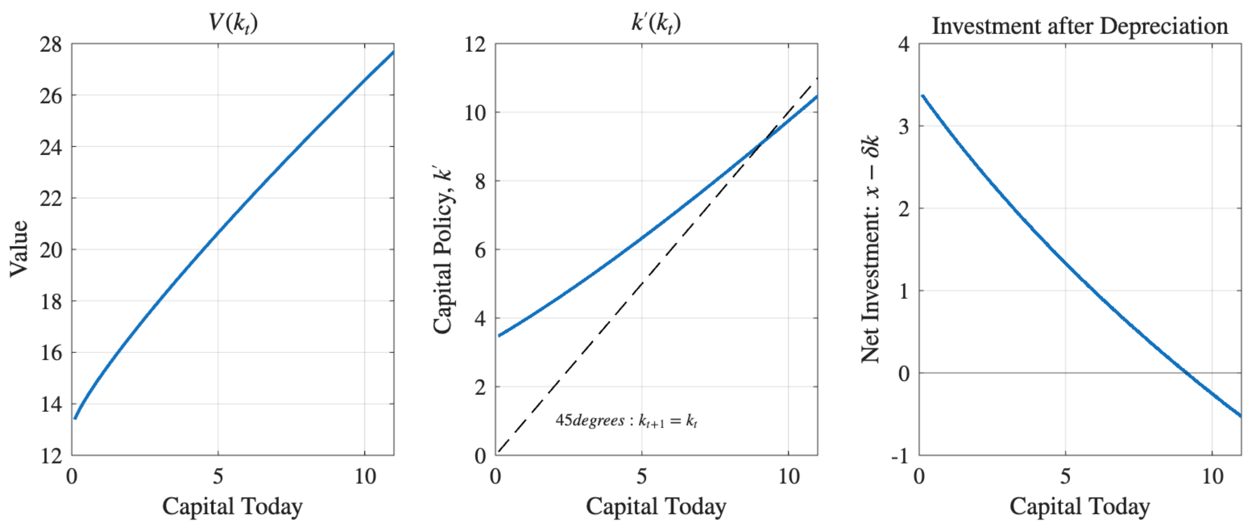

Figure 1: VFI Solution to Q Model

Optimal Investment

Solving for the value function and policies should give the following functions. Note the crossing point of the capital policy (capital for next period) across the 45 degree line. This represents the point at which $k_t = k_{t+1}$, known as the Steady State level of capital of k_{ss}

[2]Note that the guess in step 2 doesn't have to be zeros. As long as the model is well defined, the algorithm will converge no matter what guess you use. By well defined, we mean that the mapping, $\mathcal{T}V \to V$, is a contraction mapping, i.e, it satisfies Blackwell's sufficient conditions of monotonicity and discounting. You are not required to know these concepts.

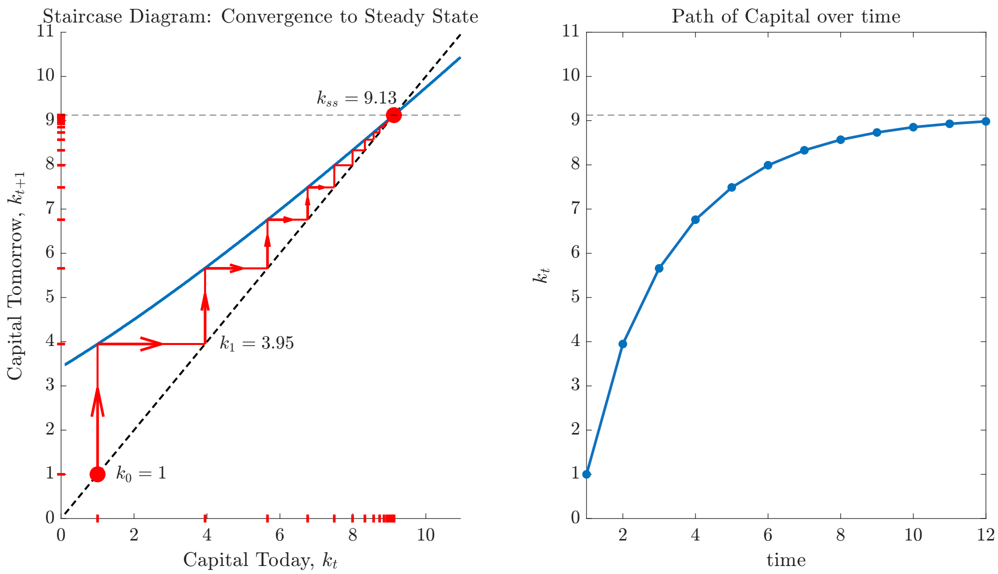

Figure 2: Convergence Path to steady state Capital

From some arbitrary value of starting capital $k_0 << k_{ss}$ we can follow convergence with the staircase plot. The chart reads as follows. From k_0 we choose some $k_1 = g(k_0)$. Next period that is our starting capital, k_1 – visually we move from the policy function at $g(k_0)$ across to the 45 degree line, this represents equality between today and tomorrow, or makes tomorrow's capital the starting point. Then we repeat. Our rule tells us to expand to $g(g(k_0))$ and then $g(g(g(k_0)))$ and so on. The right side plots the optimal path of capital from $k = 1$ towards $k_{ss} = 9.13$

How should you choose the bounds in the capital grid?

The capital grid points are set to lie in and around the steady state level of capital ($K = K'$). This is shown by the intersection of next period capital and the 45 degree line in the (K, K') plane. See below figure:

The steady state level of capital is around 9. Most points are below the steady state. We could increase the upper bound on the capital grid to have more points above the steady state. If you change the upper bound on the capital grid from 10 to 20, we get a more equitable distribution as shown below

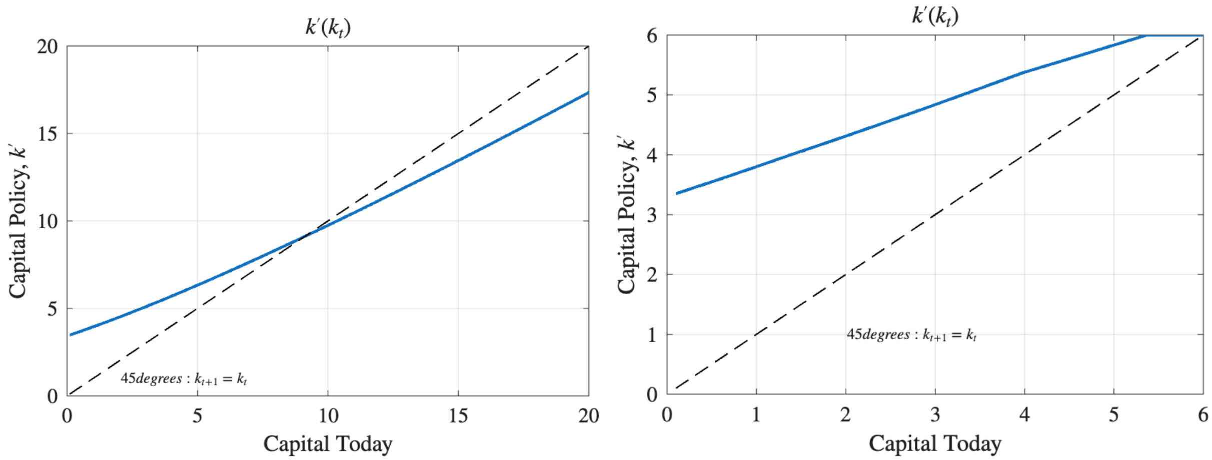

Figure 3: (a) Increasing upperbound to 20 (b) A capital grid that is too restrictive

Note that in simple deterministic models such as this one, steady states can be calculated even analytically. However, in most general models, this is not feasible. So the alternate approach in setting the bounds is to focus on the policy function. For instance, if next period capital $K'(K)$ takes values at the end points (either lower or upper bound) of the capital grid (in other words, the state space is binding) then you should relax these bounds. As the case maybe, either increase the upper bound or decrease the lower bound.

See the two figures. In the second panel, we restrict the upper bound to a low value of $K_m = 6$, well below the steady state level. As you can see, the optimal next period capital is binding at higher levels of capital. In the left panel, the optimal next period capital is no longer binding.