Question 1: Returns to Scale and Euler's Theorem

Compute $Y_K K + Y_L L$: for the functions $(i)\ Y = AK^a + BL^a \qquad (ii) Y = AK^a L^b$

Piecewise additive with curvature:

$$Y = AK^a + BL^a; \quad Y_K = aAK^{a-1} \quad Y_L = aBL^{a-1} \tag{1}$$

$$Y_K K + Y_L L = a[AK^{a-1+1}] + a[BL^{a-1+1}] \tag{2}$$

$$= aY \quad \text{For this function } Y_K K + Y_L L = Y \text{ only when } a = 1 \tag{3}$$

Cobb-Douglas:

$$Y = AK^a L^b; \quad Y_K = aAK^{a-1}L^b \quad Y_L = bAK^a L^{b-1} \tag{4}$$

$$Y_K K + Y_L L = aAK^{a-1+1}L^b + bAK^a L^{b-1+1} \tag{5}$$

$$= a[AK^a L^b] + b[AK^a L^b] \tag{6}$$

$$= (a+b)Y \quad \text{For this function } Y_K K + Y_L L = Y \text{ only when } (a+b) = 1 \tag{7}$$

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1 Q2: Minima and Maxima

(a) Find turning points of $\dfrac{x^2+1}{x}$ on the interval $[0.5, 2]$

(b) Find the maximum of $f(x) = x^2 e^{-x}$

Using quotient rule:

$$u = x^2 + 1 \qquad u' = 2x \tag{8}$$

$$v = x \qquad v' = 1 \tag{9}$$

$$(vu' - uv')/v^2 = \Big(x(2x) - (x^2+1)(1)\Big)/x^2 = (2x^2 - x^2 - 1)/x^2 = (x^2-1)/x^2 = 0 \tag{10}$$

$x^* = \pm 1$, only the positive solution is in the range. Verify this is a minimum with SOC: $f''(x) = \frac{2}{x^3}, f''(1) > 0$. This is verifed as a minimum point.

Now the maxima: the function will achieve its max on one of both of the boundaries since it is upward sloping.

$$f(0.5) = 2(1.25) = 2.5 \qquad f(2) = 5/2 = 2.5$$

Inside the stated interval, the two local maxima are the boundaries $x = (0.5, 2)$

(b) Find the maximum of $f(x) = x^2 e^{-x}$ Product rule: $(uv)' = vu' + v'u$

$$u = x^2 \qquad u' = 2x \tag{11}$$

$$v = e^{-x} \qquad v' = -e^{-x} \tag{12}$$

Combining parts: $f'(x) = 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2 - x)$ Note the term in parentheses determines the slope. So when $x = 2, f'(2) = 0$ On either side, $f(2 - \varepsilon) > 0$ and $f(2 + \varepsilon) < 0$, so slope is $(+, 0, -)$ or SOC is $(-)$, therefore $x = 2$ is a maximum.

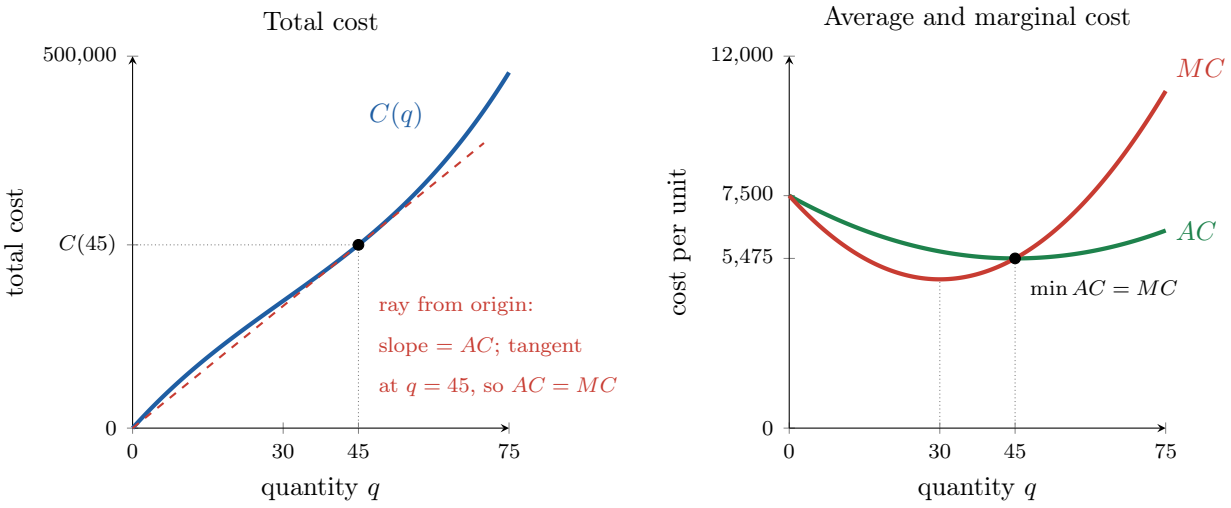
Q3: Average and Marginal Cost Functions

A firm has the total cost function

$$C(q) = q^3 - 90q^2 + 7500q \quad \text{for } q \geq 0$$

where q is quantity produced. (a) Find the average cost (cost per-unit of q) and the marginal cost for the firm. (b) Find where average cost is minimized. Compare the values of average and marginal costs at that point.

Note: A ray from the origin to point $C(q)$ will have the slope $AC(q) = C(q)/q$



Average Cost function:

$$AC(q) = \frac{C(q)}{q} = \frac{q^3 - 90q^2 + 7500q}{q} \tag{13}$$

$$= q^2 - 90q + 7500 \tag{14}$$

Marginal Cost function:

$$MC(q) = C'(q) = q^3 - 90q^2 + 7500q \tag{15}$$

$$= 3q^2 - 180q + 7500 \tag{16}$$

So we can see both MC and AC have quadratic parabola shapes, the coefficient on the x^2-term is positive for both so the U-shape is present.

Lowest $AC(q)$:

$$AC'(q) = \frac{d}{dq}(q^2 - 90q + 7500) = 2q - 90 = 0 \tag{17}$$

$$\Rightarrow q^* = 45, \tag{18}$$

$$MC(45) = 6075 - 8100 + 7500 = 5475 \tag{19}$$

$$AC(45) = 2025 - 4050 + 7500 = 5475 \tag{20}$$

In general this is a feature of cost functions. The average will keep falling while the cost of the next unit (MC) is below average. Eventually the cost of the next unit will start rising, but the average continues to fall as long as marginal cost remains below AC (here, between $q = 30$ and

$q = 45$). Once marginal cost climbs above average cost, each additional unit pulls the average up. The turning point, where average cost is at its minimum, is exactly where the two are equal.

Question 4: Profit Functions from Demand and Costs

A shipping firm sells an amount Q_A of a good in Market A with demand function

$$Q_A = 3000 - 3P_A,$$

where P_A is the price in Market A. The firm buys the amount $Q_B = Q_A$ of the same good in Market B with supply function

$$Q_B = 6P_B - 4800,$$

where P_B is the price in Market B. To transport the amount $Q = Q_A = Q_B$ from Market B to Market A, the firm pays transportation cost $100Q$.

(a) Write down the expression for the profit of the firm as a function of one variable Q.

(b) Find the amount of the good that maximises the firm's profit.

(c) Suppose that the government imposes an import tax t per unit of good that the firm must pay. Assume $t \leq 100$. Find the amount of the good that maximises the firm's profit now, as a function of the tax t.

(d) Can you advise on what tax the government should set to obtain the highest revenue from the tax if the firm chooses the level of Q which maximises its profit (your answer to part (c))? Hint: total tax revenue is $T(t) = t \cdot Q$.

Let's assume we are the only buyer in market B. We can invert the demand functions so $P_j = g(Q_j)$:

$$P_A = 1000 - \frac{1}{3}Q_A \tag{21}$$

$$P_B = 800 + \frac{1}{6}Q_A \tag{22}$$

Firm profits are given by revenue minus costs:

$$\pi(Q) = R(Q) - C(Q) = p_A(Q) \cdot Q - p_B(Q) \cdot Q - 100Q$$

, therefore, once we plug in the price functions:

$$\pi(Q) = (1000 - \frac{1}{3}Q)Q - (800 + \frac{1}{6}Q)Q - 100Q \tag{23}$$

$$= 100Q - \frac{Q^2}{2} \tag{24}$$

FOC:

$$\pi'(Q) = 100 - Q = 0 \tag{25}$$

$$\Rightarrow Q^* = 100 \tag{26}$$

Immediately we know π is a concave function, and quadratic so the FOC is also the maximum. Verify $\pi''(Q) = -1$. SOC confirms.

With the tax rate $t < 100$ the firms new profit function is then:

$$\Pi(Q, t) = 100Q - \frac{Q^2}{2} - tQ \tag{27}$$

$$= (100 - t)Q - \frac{Q^2}{2} \tag{28}$$

$$FOC : Q^* = (100 - t) \tag{29}$$

Total Revenue for government:

$$Rev(t) = t \cdot Q(t) = t(100 - t) = 100t - t^2$$

Maximising revenues (again over a concave function):

$$R'(t) = 100 - 2t : \quad t^* = 50 \tag{30}$$

Question 5: from Marginal Cost to Supply Function

A firm has the marginal cost function

$$C'(Q) = 2Q^{1/3} + 1$$

and zero fixed costs. The firm sells its output on the competitive market at price P. The firm maximises profit π.

(a) Find the firm's optimal output as a function of the price P, when $P \geq 1$.

(b) Treating the function you obtained in (a) as the firm's supply function, find the elasticity of this supply function at $P = 5$.

If we know a firm's cost structure we can compute its supply function.

$$C(q) = \int C'(q)dq$$

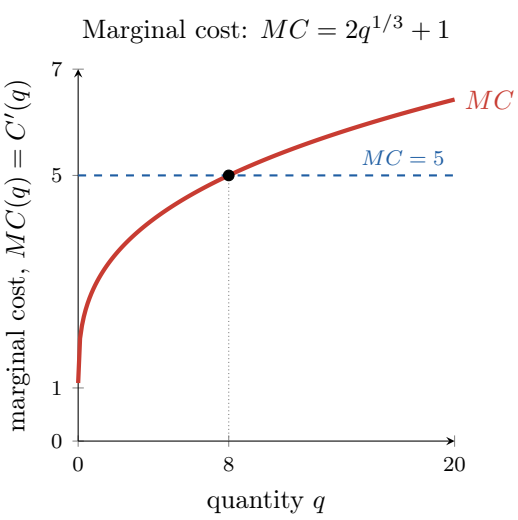
$$\int (2q^{1/3} + 1)dq = \frac{3}{2}q^{4/3} + q + FC \tag{31}$$

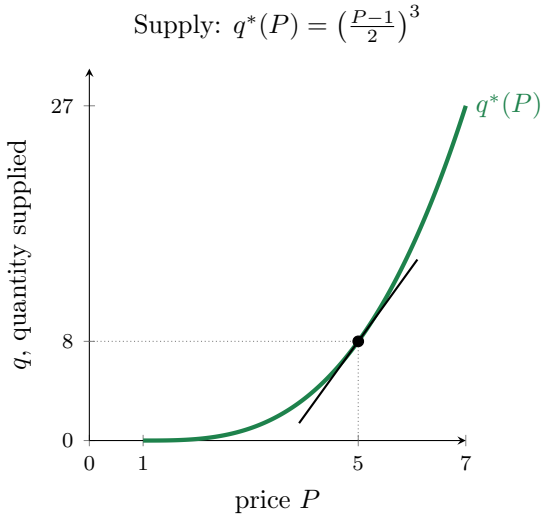
Where $FC = 0$

$$\pi(q) = Pq - C(q) = Pq - \frac{3}{2}q^{4/3} - q \tag{32}$$

FOC:

$$\pi'(q) = P - 2q^{1/3} - 1 : \qquad q^* = \left(\frac{P-1}{2}\right)^3 \tag{33}$$





Price elasticity of supply:

$$E_{q,P} = \frac{\%\text{ in } q}{\%\text{ in } P} = \frac{\frac{d\ln[q(P)]}{dP}}{\frac{d\ln P}{dP}} = \frac{q'(P)/q(P)}{1/P} = P \cdot \frac{q'(P)}{q(P)} \tag{34}$$

Staying in logs is actually easier in this case:

$$\ln q = 3\ln(P-1) - 3\ln(2), \qquad E_{q,P} = \frac{3P}{P-1}\Big|_{P=5} = 15/4 = 3.75 \tag{35}$$

Hopefully you see if we replace $MC = p$ and invert the cost function to express q in terms of P the two are mutally consistent

Question 6: Some Integrations

(a) $\int_1^2 (3x + x^{-2})dx$ (b) $\int \ln x^2 dx, x > 0$ (hint: use integration by parts for (b)

(a) is just brute-force power-rules of integration: $\int x^n = \frac{x^{n+1}}{n+1}$

$$\int_1^2 (3x + x^{-2})dx = \left[\frac{3}{2}x^2 + (-1)x^{-1}\right]_1^2 = \left[\frac{3(4)}{2} - \frac{1}{2}\right] - \left[\frac{3(1)}{2} - \frac{2}{2}\right] = \frac{11-1}{2} = 5 \tag{36}$$

(b) The hint already tells use to use integration by parts on $\ln x^2$ and 1. Recall that IBP exploits the product rule $(uv)' = vu' + uv'$

$$IBP: \int (uv)' = \int vu' + \int uv' \quad \Leftrightarrow \quad \int vu' = (uv) - \int uv' \tag{37}$$

$$v = 2\ln x, \quad v' = 2/x, \quad u = x, \quad u' = 1$$

$$\int (2\ln x)(1) = 2x\ln x - \int 2dx + C = 2x\ln x - 2x + C \tag{38}$$