Question 1: differentiation

(a) Differentiate $\dfrac{x^{1/2}-2}{x^{1/2}+1}$.

Let $u = x^{1/2} - 2$ and $v = x^{1/2} + 1$, so that $u' = v' = \frac{1}{2}x^{-1/2}$. Using the quotient rule, for $x > 0$,

$$\begin{aligned}\left(\frac{u}{v}\right)' &= \frac{vu' - v'u}{v^2} \\ &= \frac{(x^{1/2}+1)\,\frac{1}{2}x^{-1/2} - (x^{1/2}-2)\,\frac{1}{2}x^{-1/2}}{(x^{1/2}+1)^2} \\ &= \frac{\left[(x^{1/2}+1) - (x^{1/2}-2)\right]\frac{1}{2}x^{-1/2}}{(x^{1/2}+1)^2} \\ &= \frac{3}{2\,x^{1/2}\,(x^{1/2}+1)^2}.\end{aligned}$$

(b) Differentiate $x^{(2x)}$.

Note: power rule or chain rule can't be used yet, since $x = e^{\ln x}$ for $x > 0$, we have $(x)^k = (e^{\ln x})^k$, and so $x^{2x} = e^{2x \ln x}$. We also need $\dfrac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$.

If $f(x)$ is in the exponent, first lets get $f'(x)$, then build $f'(x)\exp[f(x)]$

$$\frac{d}{dx}[2x \ln x] = (2 + 2\ln x) \tag{1}$$

$$\Rightarrow \frac{d}{dx}e^{2x \ln x} = (2 + 2\ln x)e^{2x \ln x} \tag{2}$$

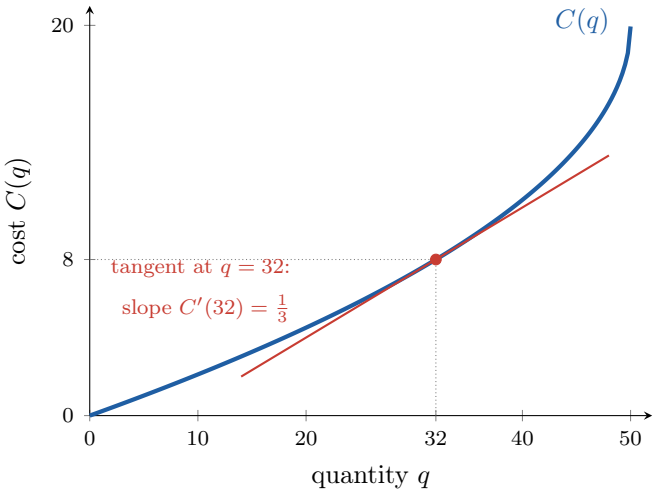
$$= 2(1 + \ln x)x^{2x} \tag{3}$$

Question 2: marginal cost

Suppose that the cost function is given by

$$C(q) = 20 - 4\left(25 - \frac{q}{2}\right)^{1/2}$$

where q is quantity produced. Find the marginal cost when $q = 32$. Hint: Think of the marginal cost as the change in the value of $C(q)$ for a very small change in q.



Marginal cost is the slope of the cost function, imagining costs were to change due to a a very small change in quantity produced. Particularly useful concept for optimality conditions which dictate at the optimum choice there should be no possible profitable deviation, no matter how small. This will usually imply some marginal benefit mins marginal cost equals zero, or more commonly, MB=MC.

$$C'(q) = (-4)(1/2)(25 - q/2)^{-1/2}(-1/2) \tag{4}$$

$$= (25 - q/2)^{-1/2} \tag{5}$$

$$C'(32) = (25 - 16)^{-1/2} \tag{6}$$

$$= 1/3 \tag{7}$$

Domain Watch out: the function in levels admits the inputs $q \in [0, 50]$, but that is not the full story. Check the marginal cost as we approach 50:

$$\text{mathematical domain: } (25 - q/2)^{1/2} \geq 0 \Rightarrow q \leq 50 \tag{8}$$

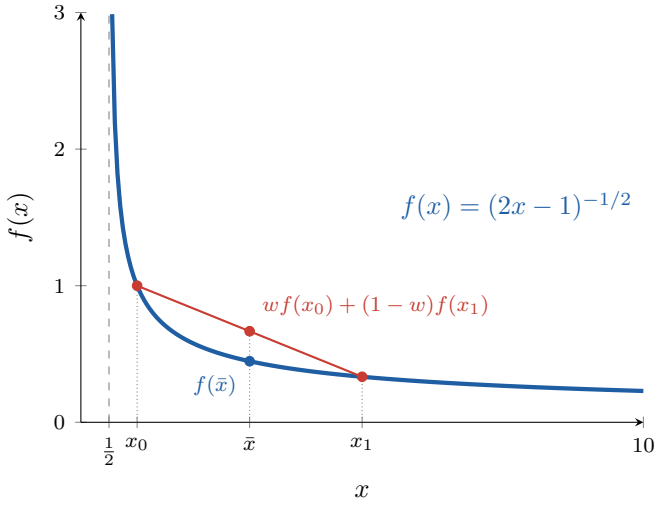
$$\text{economic domain: } \lim_{q \to 50} \{C'(q)\} = \infty \Rightarrow q < 50 \tag{9}$$

Think of 50 as a hard capacity constraint, as we approach, that last increment in production gets infinitely difficult.

Question 3: curvature, concave or convex

Is the function $f(x) = (2x-1)^{-1/2}$ concave or convex on its domain?

For a function of a single input and a single output, checking covexity/concavity is relatively straightforward (a bit more involved if $z = f(x, y)$, will see in later weeks.



$$\text{level: } f(x) = (2x-1)^{-1/2} > 0 \quad \text{defined for } x > 1/2 \tag{10}$$

$$\text{slope: } f'(x) = -(1/2)(2)(2x-1)^{-3/2} = -(2x-1)^{-3/2} < 0 \tag{11}$$

$$\text{curvature: } f''(x) = (3/2)(2)(2x-1)^{-5/2} = 3(2x-1)^{-5/2} > 0 \tag{12}$$

WE can deduce the curve is downward sloping over its full support, however the slope becomes shallower as x increases. This satisfies strict convexity:

Definition: Convexity. For any weight w between 0 and 1 (0 to 100 percent), and any two distinct points x_0, x_1:

$$w \cdot f(x_0) + (1-w) \cdot f(x_1) > f\big(w \cdot x_0 + (1-w) \cdot x_1\big)$$

$$\text{average of } f \;>\; f(\text{average})$$

The average of the function (evaluated at two points) is strictly above the function evaluated at an average of the two input points. Drawn in red above.

	Convex (V-shape)	**Concave** (C-shape)
Chords	above $f(x)$	below $f(x)$
Slope, $f'(x)$	rises with x	falls with x
Second derivative	$f''(x) > 0$ for all x	$f''(x) < 0$ for all x
Curvature	bends up	bends down
At a stationary point	local minimum	local maximum
Examples	x^2	$\sqrt{x}$

Question 4: elasticity, proportional responses of outputs to inputs

(a) Prove that if $F(x) = \dfrac{f(x)}{g(x)}$, then $\dfrac{F'(x)}{F(x)} = \dfrac{f'(x)}{f(x)} - \dfrac{g'(x)}{g(x)}$.

Assume f and g are differentiable with $f(x) \neq 0$ and $g(x) \neq 0$. By the quotient rule,

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Dividing by $F(x) = f(x)/g(x)$, is the same as multiplying by $g(x)/f(x)$

$$\begin{aligned}\frac{F'(x)}{F(x)} &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \cdot \frac{g(x)}{f(x)} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{f(x)\,g(x)} \\ &= \boxed{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}}\end{aligned}$$

(b) Using your result from part (a), show that the elasticity $\varepsilon_{F,x} = \varepsilon_{f,x} - \varepsilon_{g,x}$.

Definition. The elasticity of a function $h(x)$ with respect to x is the percentage change in $h(x)$ per one percent change in x:

$$\varepsilon_{h,x} = \frac{\%\text{ in } h}{\%\text{ in } x} = \frac{\frac{d\ln[h(x)]}{dx}}{\frac{d\ln x}{dx}} = \frac{\frac{d\ln h}{dh}\frac{dh}{dx}}{1/x} = \frac{h'(x)/h(x)}{1/x} = \boxed{x \cdot \frac{h'(x)}{h(x)}}$$

Elasticity is about proportional responsiveness

Consider the two functions, and their elasticities:

$$(i)\ D(p) = \alpha + \beta p \qquad (ii)\ f(k) = zk^{\gamma}$$

Application: Strait of Hormuz Closure Impact on Oil Price

Research done up until 2019 suggest the price elasticity of oil demand is around -0.3

$$\text{price elasticity of oil demand: } \varepsilon_{q,p} = \frac{\%\text{ in quantities of oil demanded}}{\%\text{ in oil price}} \approx -0.33$$

is very small in magnitude and negative, meaning changes in the oil price do not move quantity demanded very much, or to paraphrase, very large increases in price are needed to reduce demand. This is natural given it is hard to substitute oil/energy in a lot of economic activity.

Can we predict price movements with the elasticity estimates? Closure of the strait of Hormuz to shipping was set to reduce work oil production by roughly 20 percent, $q_{after} = 0.8q_{before}$ or

$$\ln q_{after} = \ln q_{before} + \ln(0.8); \quad \Delta \ln q = \ln(0.8)$$

. How much do prices need to adjust to match demand to the reduced global oil supply:

Hormuz Back-of-the-envelope prediction (demand only):

$$\Delta \ln p = \frac{\Delta \ln q}{\varepsilon_d} = \frac{\ln(0.8)}{-0.33} = \frac{-0.22}{-0.33} \approx 0.68; \quad \%\Delta p = e^{0.68} - 1 \approx 0.97 \boxed{\approx +100\%}$$

Adjust for supply response, Baumeister and Hamilton (2019), $\varepsilon_s \approx 0.15$:

$$\Delta \ln p = \frac{\ln(0.8)}{(\varepsilon_d - \varepsilon_s)} = \frac{-0.22}{-0.15 - 0.33} \approx 0.46; \quad \%\Delta p = e^{0.46} - 1 \approx 0.59 \boxed{\approx +60\%}$$

Hormuz in the data (WTI oil price, FRED):

- Pre-closure lows: Dec 2025, around 58 USD/barrel
- Crisis avg: Mar–Oct 2026, 93 USD/b, i.e. $\Delta \ln p = \ln(93/58) \approx 0.47; e^{0.47} - 1$ $\boxed{+60\%}$

Applying the definition: the three elasticities in the question are

$$\varepsilon_{F,x} = x\,\frac{F'(x)}{F(x)}, \qquad \varepsilon_{f,x} = x\,\frac{f'(x)}{f(x)}, \qquad \varepsilon_{g,x} = x\,\frac{g'(x)}{g(x)}.$$

Part (a) showed that

$$\frac{F'(x)}{F(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Multiplying both sides by x,

$$x\,\frac{F'(x)}{F(x)} = x\,\frac{f'(x)}{f(x)} - x\,\frac{g'(x)}{g(x)},$$

and each term is an elasticity, so

$$\varepsilon_{F,x} = \varepsilon_{f,x} - \varepsilon_{g,x}.$$

Interpretation of result. for each percent change in x, f responds by $(\varepsilon_{f,x})\%$ and g by $(\varepsilon_{g,x})\%$, then the ratio $F = f/g$ changes by approximately $(\varepsilon_{f,x} - \varepsilon_{g,x})\%$: the percentage change in a ratio is the percentage change in the numerator minus the percentage change in the denominator. We like elasticities because they are unitless

Question 5: income elasticity of demand

Demand $D(I) = AI^{1.23}$. How does demand change for a percentage change in income?

Fairly straight forward:

$$\varepsilon(D, I) = I \cdot \frac{D'(I)}{D(I)} = \frac{(1.23)AI^{1.23-1}I}{AI^{1.23}} = 1.23 \tag{13}$$

for a 1 percent increase in income will lead to a 1.23 percent increase in demand, approximately.

Interpretation: If elasticity is greater than 1 we call demand "elastic", quantity demanded responds more than one-for-one on a proportional basis to a change income. If elasticity is less than 1 we call demand "inelastic".