Summary of Key Objects

- **Vector**: 3 interpretations: (1) direction and magnitude, (2) a coordinates system of points in space, (3) an item on an abstract list of n varieties

- **Matrix**: geometric interpretation as a linear transformation of space: a stretching / squashing /rotation /shear of space that keeps grid lines parallel and evenly spaced and fixes the origin.

  Simple interpretation: representation of a linear system of equations.

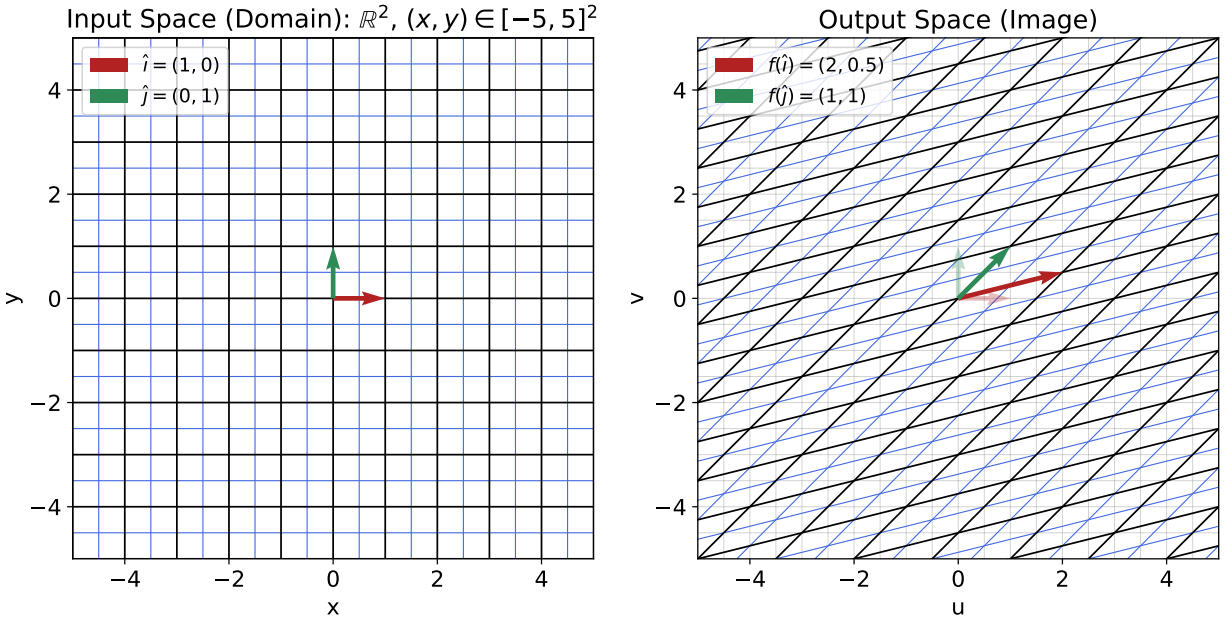


Figure 1: A linear transformation

- **Linear Transformation as a Matrix:** A transformation L is *linear* iff (not a typo: if and only if), for all vectors $\mathbf{u}, \mathbf{v}$ and all scalars α:

  1. $L(\mathbf{v} + \mathbf{u}) = L(\mathbf{v}) + L(\mathbf{u})$
  2. $L(\alpha\mathbf{v}) = \alpha L(\mathbf{v})$

  It turns out the transformations $L : \mathbb{R}^n \to \mathbb{R}^m$ that satisfy these requirements have to look like this:

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

  and rules out things like (e.g.) $f([x, y]') = (x + \sin y, y + \sin x)'$. Which make space very wavy.

Sticking only to linear transformations of the space, $L(\mathbf{v}) = A\mathbf{v}$. We can summarise the transformation simply by tracking where A sends the basis vectors (unit steps East and North):

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = L\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x \cdot L\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \cdot L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} L(\hat{i}) & L(\hat{j}) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

For example, $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

- will send the East step $\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
- will send the North step $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- any other points in the space can be shown to be a combo of these steps put together (lateral plus vertical)

- **Span**: The span of **A** is the set of all possible points we can reach with some linear combination of the columns of **A**. So when the vectors are not lined up:

$$\text{span}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \left\{\begin{bmatrix} u \\ v \end{bmatrix} : \begin{bmatrix} u \\ v \end{bmatrix} = \alpha\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for any scalars } \alpha, \beta\right\} = \mathbb{R}^2, \text{the full 2D plane}$$

and when the columns overlap, we are stuck on the line, we can't move laterally

$$span\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \{(x, y) : y = 0.5x\}$$

- **Determinant**: related to rescaling area after transformation. Consider the unit-square made by $(\hat{i}, \hat{j})$ versus $L(\hat{i}), L(\hat{j})$. The determinant is the ratio of the two areas.

- $\det(A) = 0$ means transformation **A** sends the space to a lower dimension (e.g. 3 (cube) to 2 (plane)) and therefore cannot be inverted: a many-to-one problem

- **Order of columns or rows matters:** $\det([u, v]) = -\det([v, u])$ swapping rows (columns) means you need to change sign when using cofactors to keep track of the overall determinant

- **Eigenvector**: for a given transformation **A**, some vector(s) will not get knocked off their own span, they remain pointing along their span, scaled by some multiplier λ. These special vectors are the eigenvectors of **A** will solve (there can be more than one, hence subscript):

$$\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i \Rightarrow (\mathbf{A} - \lambda_i\mathbb{I}_n)\mathbf{x}_i = \mathbf{0}_n \Rightarrow \det(\mathbf{A} - \lambda_i\mathbb{I}_n) = 0$$

so we need matrix $(A - \lambda_i\mathbb{I}_n)$ to reduce space to a lower dimension, hence the determinant condition.

- **Eigendecomposition (diagonalisation)**: if A has a full set of linearly independent eigenvectors, stack them as the columns of X and put the matching eigenvalues, in the same order, on the diagonal of D. Then

$$\mathbf{AX} = \mathbf{XD} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{XDX}^{-1}.$$

For example, with $\lambda_1 = 3$, $\mathbf{v}_1 = (1, 2)'$ and $\lambda_2 = 0.5$, $\mathbf{v}_2 = (2, -1)'$:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2.5 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}_{X=[\mathbf{v}_1\ \mathbf{v}_2]} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}}_{D} \underbrace{\tfrac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}_{X^{-1}}.$$

From the above you can easily show (check $\mathbf{A}^2$) that $\mathbf{A}^n = \mathbf{XD}^n\mathbf{X}^{-1}$ with $\mathbf{D}^n = \begin{bmatrix} 3^n & 0 \\ 0 & 0.5^n \end{bmatrix}$. **Eigenvalues control whether repeated matrix multiplication is convergent or explosive.**

- **Relationship Eigenvalues-Determinant** Since Eigenvalues and determinants relate to scale, it is natural they are related. If $det(X) = k$ imagine this scales up area by factor k, then it is natural the inverse scales space down again with factor $1/k$. In the following product, the two determinants cancel out:

$$det(A) = det(\mathbf{XDX}^{-1}) = det(\mathbf{X}) \cdot det(\mathbf{D}) \cdot det(\mathbf{X}^{-1}) = det(\mathbf{D})$$

In my 2D example $det(A) = det\begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix} = 3(0.5) = 1.5$. Intuitively we are scaling by 3 in one direction (v_1) and 0.5 in the other (v_2), so overall area will scale by the product of both.

- **Cramer's Rule**: exploit the geometry of parallelograms and determinants (Area $= b \times h$) to solve linear equation systems, but can solve one variable alone.

$$x_j^* = \frac{det(D_j)}{det(A)}; D_j \text{ replaces } A\text{'s jth-column with } b$$

- **Determinant by cofactor expansion**: pick any one row or column (ideally the one with the most zeros) and expand:

$$|A| = \sum_j a_{ij}\, C_{ij}, \qquad C_{ij} = (-1)^{i+j} M_{ij},$$

where the minor M_{ij} is the determinant of A with row i and column j deleted. The signs follow the checkerboard $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$. For 3×3 only, Sarrus' rule is a shortcut.

- **Inverse**: A^{-1} exists if and only if $|A| \neq 0$, and

$$A^{-1} = \frac{1}{|A|}\,\text{adj}(A), \qquad \text{adj}(A) = C',$$

i.e. the *transpose* of the matrix of cofactors. For a 2×2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Check by multiplying back: $AA^{-1} = I$.

- **Gaussian Elimination**: solve the system with row operations (equation-by-equation, e.g. row 3 - row 1) to get a triangular matrix, then back-substitute for the solution.

$$\mathbf{Ax} = \mathbf{b}; \qquad [\mathbf{A} \,|\, \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 1 & 11 \\ 1 & 2 & 3 & 14 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 9 \end{array}\right] \Rightarrow z = 3,\ y = 2,\ x = 1$$

- **Input–Output Table/Matrix**: the matrix associated with flows of goods along production networks. Element a_{ij} represents flows of goods from sector $i \rightarrow j$. So columns are recipes.

- **Leontief Inverse Matrix**: a very important matrix in macroeconomics. Reveals total effects through direct and indirect effects along supply chains. Small shocks can have large effects if they trigger lots of chain reactions from very central nodes. Think: energy. A tiny share of GDP, but if we shut down electricity we would destroy the economy. For a vector of shocks $\mathbf{s} = d\mathbf{D}$:

$$\mathbf{dX} = (I - T)^{-1}\mathbf{s}$$

Question 1: matrix algebra

Question 1

Let

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & -1 \end{bmatrix}.$$

(a) Find $(ABC)' + 2(C'B)$. (b) Find $(CC')^2$, where $(CC')^2 = (CC')(CC')$.

Before you multiply: check dimensions

A product XY exists only if (columns of X) = (rows of Y), and the result has the outer dimensions:

$$\underbrace{A}_{2\times 2}\ \underbrace{B}_{2\times 2}\ \underbrace{C}_{2\times 3} \to\ 2\times 3, \qquad (ABC)':\ 3\times 2, \qquad \underbrace{C'}_{3\times 2}\ \underbrace{B}_{2\times 2} \to\ 3\times 2.$$

Both terms are 3×2, so the sum is defined and the answer must be 3×2. Similarly CC' is $(2 \times 3)(3 \times 2) = 2 \times 2$.

(a) Start with AB:

$$AB = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2(-2)+4(1) & 2(4)+4(-2) \\ 1(-2)+2(1) & 1(4)+2(-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $ABC = 0_{2\times 3}$ and $(ABC)' = 0_{3\times 2}$. Next,

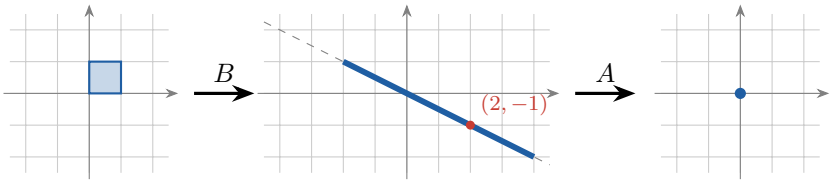
$$C'B = \begin{bmatrix} 1 & -1 \\ -1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 5 & -10 \\ -5 & 10 \end{bmatrix}, \qquad 2(C'B) = \begin{bmatrix} -6 & 12 \\ 10 & -20 \\ -10 & 20 \end{bmatrix}.$$

Therefore

Answer. $(ABC)' + 2(C'B) = \begin{bmatrix} -6 & 12 \\ 10 & -20 \\ -10 & 20 \end{bmatrix}$.

Why is $AB = 0$ when neither A nor B is zero?

With real numbers, $ab = 0$ forces $a = 0$ or $b = 0$. With matrices it does not. Think of each matrix as a *map* of the plane. The columns of B, $(-2, 1)'$ and $(4, -2)'$, are both multiples of $(2, -1)'$, so B squashes the whole plane onto the line through $(2, -1)'$. But $A(2, -1)' = (0, 0)'$: that line is exactly the set of vectors A sends to zero (the *null space* of A). So A annihilates everything B produces.



Corollary: both A and B are singular ($|A| = 2 \cdot 2 - 4 \cdot 1 = 0$, $|B| = 4 - 4 = 0$), since a map that collapses the plane onto a line destroys area (see the determinant note in Question 2).

Check: a shortcut for (a). Using $(XYZ)' = Z'Y'X'$ you could also write $(ABC)' = C'B'A'$, but spotting $AB = 0$ first saves all of that work: look for structure before brute-force multiplication.

(b)

$$CC' = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix},$$

$$(CC')^2 = \begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix} \begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix} = \begin{bmatrix} 36+36 & -36-66 \\ -36-66 & 36+121 \end{bmatrix} = \begin{bmatrix} 72 & -102 \\ -102 & 157 \end{bmatrix}.$$

Answer. $(CC')^2 = \begin{bmatrix} 72 & -102 \\ -102 & 157 \end{bmatrix}$.

Check: CC' is always *symmetric*, because $(CC')' = (C')'C' = CC'$; so is any power of it. Its diagonal entries are the squared lengths of the rows of C: $1^2 + (-1)^2 + 2^2 = 6$ and $(-1)^2 + 3^2 + (-1)^2 = 11$. Off-diagonal: the dot product of the two rows, $-1 - 3 - 2 = -6$. If your CC' or $(CC')^2$ is not symmetric, there is an arithmetic slip.

Note: $(CC')^2$ is *not* obtained by squaring each entry $(36, 36, 36, 121)$. Matrix powers are repeated matrix products.

Question 2: determinants and inverses

Question 2

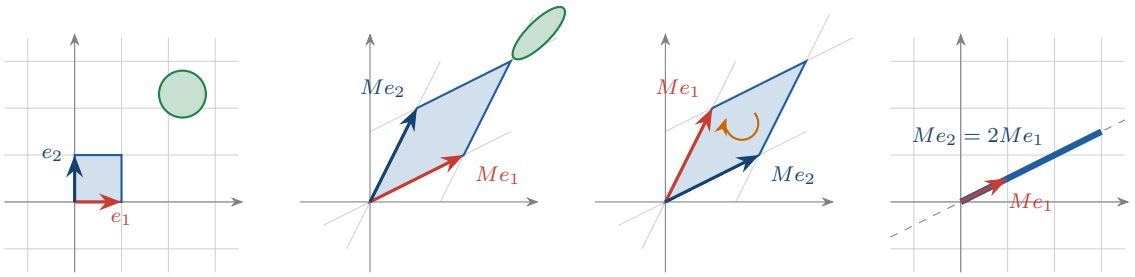
Let $A_t = \begin{bmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & -1 & 2 \end{bmatrix}$, where t is a parameter. (a) Find the determinant $|A_t|$. (b) For what values of t does A_t have an inverse? (c) Find the inverse A_t^{-1} when $t = 0$.

What a determinant *is*: the area (volume) scaling factor

A square matrix M is a linear map $x \mapsto Mx$. It sends the unit square (spanned by e_1, e_2) to the parallelogram spanned by its *columns* Me_1, Me_2, and

$$\text{abs}(\det M) = \frac{\text{area of the output shape}}{\text{area of the input shape}} \qquad \text{for } \textit{any} \text{ input shape (square, disc, ...)}.$$

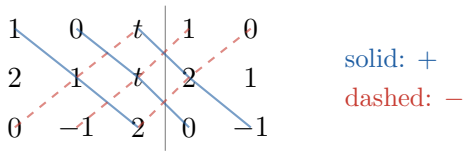
The *sign* records orientation: $\det M < 0$ means the map flips the plane over (the images of e_1, e_2 come in clockwise order). In three dimensions, "area" becomes "volume".



(i) input area 1 | (ii) $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ area $\times 3$ | (iii) $M = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ area $\times 3$, flipped: $\det = -3$ | (iv) A from Q1: $\left(\begin{smallmatrix} 2 & 4 \\ 1 & 2 \end{smallmatrix}\right)$ area $\times 0$: collapse

Panel (ii) maps the green disc of area $\pi(0.25)^2$ into an ellipse of area exactly $3 \times \pi(0.25)^2$: the factor does not depend on the shape. Panel (iii) uses the same columns in swapped order: same area, opposite orientation, so swapping two columns flips the sign of a determinant. Panel (iv) is why $\det = 0 \iff$ no inverse: once the plane is crushed onto a line, there is no way to "un-crush" it, because many inputs share the same output.

(a) For a 3×3 matrix we can use the rule of Sarrus (copy the first two columns to the right; add the three "down-right" products, subtract the three "up-right" products):



$$|A_t| = \underbrace{1 \cdot 1 \cdot 2 + 0 \cdot t \cdot 0 + t \cdot 2 \cdot (-1)}_{\text{down-right}} - \underbrace{\big(0 \cdot 1 \cdot t + (-1) \cdot t \cdot 1 + 2 \cdot 2 \cdot 0\big)}_{\text{up-right}} = 2 - 2t - (-t) = 2 - t.$$

Answer. $|A_t| = 2 - t$.

Check: same answer by cofactor expansion along row 1. $|A_t| = 1 \cdot \begin{vmatrix} 1 & t \\ -1 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & t \\ 0 & 2 \end{vmatrix} + t \cdot \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = (2 + t) - 0 + t(-2) = 2 - t.$ ✓

Note: Sarrus' rule works *only* for 3×3 matrices. For 4×4 (Questions 3 and 4) you must use cofactor expansion or row reduction. **Note:** CF expansion using the same (+,-,+) alternating pattern in odd rows, colums and (-,+,-) in even.

(b) A_t is invertible if and only if $|A_t| \neq 0$, i.e. $2 - t \neq 0$.

Answer. A_t^{-1} exists if and only if $t \neq 2$.

(c) At $t = 0$, $A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$ and $|A_0| = 2$. The inverse is $A_0^{-1} = \frac{1}{|A_0|} \operatorname{adj}(A_0)$, where the

adjugate $\operatorname{adj}(A_0)$ is the *transpose* of the matrix of cofactors $C_{ij} = (-1)^{i+j} M_{ij}$ (signs follow the checkerboard $\left(\begin{smallmatrix} + & - & + \\ - & + & - \\ + & - & + \end{smallmatrix}\right)$):

$$C = \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ -1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 2 & -4 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \operatorname{adj}(A_0) = C' = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix}.$$

Answer. $A_0^{-1} = \dfrac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$

Check: multiply back: $A_0 A_0^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2-2 & 1 & 0 \\ 2-2 & -1+1 & 1 \end{bmatrix} = I_3$. ✓

Two further checks. (i) A_0 is lower triangular, so $|A_0|$ is the product of the diagonal, $1 \cdot 1 \cdot 2 = 2$, and its inverse must also be lower triangular. (ii) $|A_0^{-1}| = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2} = 1/|A_0|$: if A_0 doubles volumes, undoing it must halve them.

Note: Remember to *transpose* the cofactor matrix. Here C and C' differ, and using $C/|A_0|$ gives an upper-triangular matrix that fails the check above.

Question 3: a macro model in matrix form

Question 3

An economy is described by

$$Y = C + I - S, \qquad C = 0.75Y + 80, \qquad I = \alpha Y + 20, \qquad S = 0.2(C + I) - 24,$$

where Y is national income, C consumption, I investment, S savings, and α is a parameter. An equilibrium is where all four equations hold. (a) Write the relationships in matrix form. (b) For what value of α is there not a unique equilibrium?

(a) Put all endogenous variables on the left, in a fixed order (Y, C, I, S), and constants on the right:

$$\begin{aligned} Y \;-\; C \;-\; I \;+\; S &= 0 \\ -0.75Y \;+\; C &= 80 \\ -\alpha Y \;+\; I &= 20 \\ 0.2C \;+\; 0.2I \;-\; S &= 24 \end{aligned}$$

Answer. $Ax = b$ with $A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -0.75 & 1 & 0 & 0 \\ -\alpha & 0 & 1 & 0 \\ 0 & 0.2 & 0.2 & -1 \end{bmatrix}$, $x = \begin{bmatrix} Y \\ C \\ I \\ S \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 80 \\ 20 \\ 24 \end{bmatrix}$.

Note: Watch the signs when moving terms across. E.g. $S = 0.2(C+I) - 24$ becomes $0.2C + 0.2I - S = 24$ (or equivalently $-0.2C - 0.2I + S = -24$; either is fine as long as you are consistent). Also keep the zero entries explicit; they matter for the determinant.

(b) A unique solution exists iff $|A| \neq 0$. Expand along row 2 (it has two zeros):

$$|A| = -(-0.75)\underbrace{\begin{vmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 0.2 & 0.2 & -1 \end{vmatrix}}_{=1-0.2=0.8} + 1 \cdot \underbrace{\begin{vmatrix} 1 & -1 & 1 \\ -\alpha & 1 & 0 \\ 0 & 0.2 & -1 \end{vmatrix}}_{=-1-0.2\alpha+\alpha=-1+0.8\alpha} + 0 + 0.$$

(The cofactor signs along row 2 are $-, +, -, +$. Each 3×3 minor is computed by Sarrus: for the first, the down-right products sum to 1 and the up-right ones to 0.2; for the second, they sum to $-1 - 0.2\alpha$ and $-\alpha$.) Therefore

$$|A| = 0.75(0.8) + (-1 + 0.8\alpha) = 0.8\alpha - 0.4 = 0.4(2\alpha - 1).$$

This is zero iff $\alpha = \frac{1}{2}$.

Answer. There is no unique equilibrium when $\alpha = \frac{1}{2}$. (exclude nonsense values of Y)

Question 4: two interrelated markets and Cramer's rule

Question 4

Market 1: demand $Q_1 = 60 - P_1 - 0.5P_2$, supply $Q_1 = 2P_1 - P_2$. Market 2: demand $Q_2 = 80 - P_1 - 2P_2$, supply $Q_2 = -3P_1 + 5P_2$. An equilibrium is when supply equals demand in each market. (a) Write the equations in matrix form. (b) Find the equilibrium price in market 1 using Cramer's rule. (c) Find the equilibrium price in market 2 using Cramer's rule.

(a) With variable order (Q_1, Q_2, P_1, P_2):

$$\begin{array}{rrrrcr} Q_1 & & + P_1 & + 0.5P_2 & = & 60 \\ Q_1 & & - 2P_1 & + P_2 & = & 0 \\ & Q_2 & + P_1 & + 2P_2 & = & 80 \\ & Q_2 & + 3P_1 & - 5P_2 & = & 0 \end{array}$$

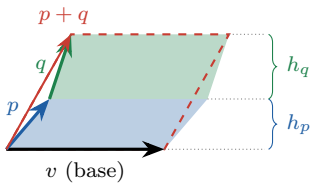
Answer. $Ax = b$ with $A = \begin{bmatrix} 1 & 0 & 1 & 0.5 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & -5 \end{bmatrix}$, $x = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix}$, $b = \begin{bmatrix} 60 \\ 0 \\ 80 \\ 0 \end{bmatrix}$.

Cramer's rule: For $Ax = b$ with $|A| \neq 0$: $x_j = |D_j|/|A|$, where D_j is A with column j replaced by b. The rest of this note shows *why*, in the 2×2 case.

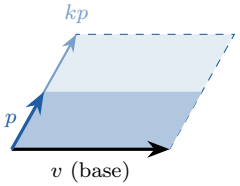
Notation. Write $\det[u,\, v]$ for the determinant of the matrix whose columns are the vectors u and v: the (signed) area of the parallelogram with sides u and v. If A has columns a_1, a_2, then

$$|A| = \det[a_1,\, a_2], \qquad |D_1| = \det[b,\, a_2], \qquad |D_2| = \det[a_1,\, b].$$

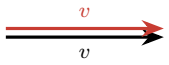
Step 1: three rules of area. Because a determinant is an area, it obeys "area = base × height". Hold one side v fixed as the base and vary the other side:



R1. Heights add



R2. Stretch a side by k



R3. Same side twice: flat

R1: $\det[p+q,\ v] = \det[p, v] + \det[q, v]$ **R2:** $\det[kp,\ v] = k\det[p, v]$ **R3:** $\det[v,\ v] = 0$

Each is a one-line check from $\det[u, v] = u_1v_2 - u_2v_1$; for instance R1 is $(p_1 + q_1)v_2 - (p_2 + q_2)v_1 = (p_1v_2 - p_2v_1) + (q_1v_2 - q_2v_1)$. R1 and R2 together say that a determinant is *linear in each column separately*, holding the other columns fixed. That is what lets us take a determinant apart piece by piece. (The same rules hold with the base in the first slot and the varying side in the second.) NB Determinant is order specific $det[u, v] = -det[v, u]$

Step 2: read the system as a recipe for b. $Ax = b$ says $x_1a_1 + x_2a_2 = b$: the unknowns are the weights on the columns.

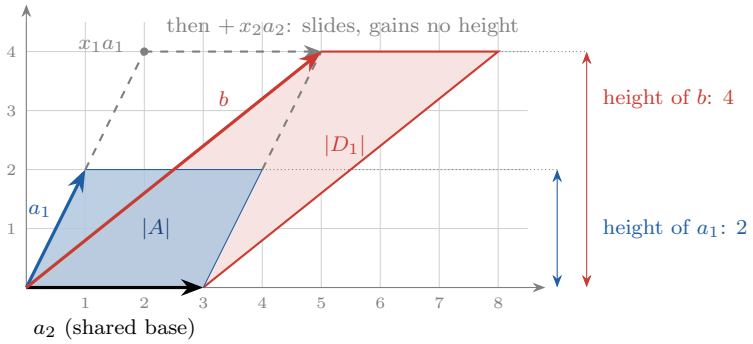
Step 3: take $|D_1|$ apart.

$$\begin{aligned} |D_1| = \det[b,\ a_2] &= \det[x_1a_1 + x_2a_2,\ a_2] && \text{write } b \text{ using Step 2} \\ &= \det[x_1a_1,\ a_2] + \det[x_2a_2,\ a_2] && \text{R1: split the sum} \\ &= x_1\det[a_1,\ a_2] + x_2\det[a_2,\ a_2] && \text{R2: pull out the weights} \\ &= x_1\det[a_1,\ a_2] = x_1|A| && \text{R3: the second term is flat.} \end{aligned}$$

Putting b in column 1 is designed to do exactly this: the x_2 piece lands next to a copy of a_2 and vanishes, so we never need to know x_2.

Step 4: divide. $x_1 = |D_1|/|A|$, which needs $|A| \neq 0$. Putting b in column 2 instead kills the x_1 piece: $|D_2| = \det[a_1,\ x_1a_1 + x_2a_2] = x_1\det[a_1, a_1] + x_2\det[a_1, a_2] = x_2|A|$, so $x_2 = |D_2|/|A|$.

The same argument as a picture. Take $a_1 = (1,2)'$, $a_2 = (3,0)'$, $b = (5,4)'$. The base a_2 lies along the horizontal axis, so "height" is just the vertical coordinate.



The grey dashed outline is the blue parallelogram stretched to sides x_1a_1 and a_2: x_1 times the height, so x_1 times the area (R2). Sliding its top edge along by x_2a_2 turns it into the red parallelogram without changing base or height, so red and grey have the same area (R1 and R3). All three stand on the same base, so the ratio of their areas is the ratio of their heights. To reach b we go x_1 steps of a_1 (each gains height 2) and then x_2 steps of a_2 (which gain none), so b's height is $x_1 \times 2$. Hence $x_1 = 4/2 = 2$. In determinants (colour-coded by column):

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, \qquad \mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$|A| = \det[\mathbf{a}_1, \mathbf{a}_2] = \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6$$

$$|D_1| = \det[\mathbf{b}, \mathbf{a}_2] = \begin{vmatrix} 5 & 3 \\ 4 & 0 \end{vmatrix} = -12, \qquad |D_2| = \det[\mathbf{a}_1, \mathbf{b}] = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = -6$$

$$x_1^* = \frac{|D_1|}{|A|} = \frac{-12}{-6} = 2, \qquad x_2^* = \frac{|D_2|}{|A|} = \frac{-6}{-6} = 1$$

Check: $2(1,2)' + 1(3,0)' = (5,4)'$. ✓ (All three determinants are negative because the sides come in clockwise order; the signs cancel in the ratios.)

Note: R1 and R2 apply to a single column with the others held fixed. They are *not* rules for whole matrices: in general $\det(A+B) \neq \det A + \det B$, and $\det(kA) = k^n \det A$ for an $n \times n$ matrix (multiplying the whole matrix by k stretches every side, so R2 applies n times; doubling both sides of a parallelogram quadruples its area).

(b) First $|A|$, expanding along column 1:

$$|A| = +1 \begin{vmatrix} 0 & -2 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & -5 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 0.5 \\ 1 & 1 & 2 \\ 1 & 3 & -5 \end{vmatrix} = 1 \cdot (-12) - 1 \cdot (8) = -20,$$

(by Sarrus: $0 - 4 + 3 - 1 - 0 - 10 = -12$ and $0 + 2 + 1.5 - 0.5 - 0 + 5 = 8$). P_1 is the 3rd variable,

so replace column 3 of A by b and expand along that column:

$$|D_3| = \begin{vmatrix} 1 & 0 & 60 & 0.5 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 80 & 2 \\ 0 & 1 & 0 & -5 \end{vmatrix} = +60 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -5 \end{vmatrix} + 80 \begin{vmatrix} 1 & 0 & 0.5 \\ 1 & 0 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 60(-7) + 80(-0.5) = -460.$$

The signs of the cofactors in column 3 are $+, -, +, -$; the 2nd and 4th entries are zero. Remember even rows/columns would reverse the sign pattern.

$$P_1 = \frac{|D_3|}{|A|} = \frac{-460}{-20} = 23.$$

Answer. $P_1 = 23$.

(c) P_2 is the 4th variable; replace column 4 by b and expand along it (cofactor signs $-, +, -, +$):

$$|D_4| = \begin{vmatrix} 1 & 0 & 1 & 60 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 80 \\ 0 & 1 & 3 & 0 \end{vmatrix} = (-)60 \begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{vmatrix} (-)80 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{vmatrix} = -60(2) - 80(3) = -360,$$

$$P_2 = \frac{|D_4|}{|A|} = \frac{-360}{-20} = 18.$$

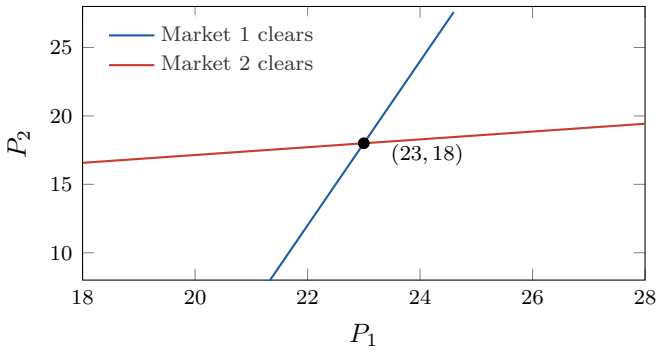
Answer. $P_2 = 18$.

Check: plug back in, and see it in price space. Quantities: $Q_1 = 2(23) - 18 = 28 = 60 - 23 - 9$ ✓; $Q_2 = -69 + 90 = 21 = 80 - 23 - 36$ ✓.

Eliminating Q_1, Q_2 (set demand = supply) leaves two market-clearing lines in the (P_1, P_2) plane:

$$\text{Market 1: } 3P_1 - 0.5P_2 = 60, \qquad \text{Market 2: } 2P_1 - 7P_2 = -80.$$

Their 2×2 determinant is $3(-7) - (-0.5)(2) = -20$, the same as $|A|$. This is no accident: subtracting supply rows from demand rows (which does not change a determinant) and expanding along the Q-columns reduces the 4×4 determinant to $\pm$ this 2×2 one. Cramer in 2D: $P_1 = \begin{vmatrix} 60 & -0.5 \\ -80 & -7 \end{vmatrix} / (-20) = -460/-20 = 23$ and $P_2 = \begin{vmatrix} 3 & 60 \\ 2 & -80 \end{vmatrix} / (-20) = -360/-20 = 18$. ✓



$|A| \neq 0 \Leftrightarrow$ the two lines are not parallel $\Leftrightarrow$ a unique crossing.

Note: Cramer's rule gives x_j for the variable in the *position* you replace. Here the variables are ordered (Q_1, Q_2, P_1, P_2), so P_1 is column 3, not column 1.

Question 5: a Leontief input–output economy

Question 5

An economy has an agricultural sector (A), an industrial sector (I), and a service sector (S). Producing 20 units of A requires 2 units of A and 5 of I. Producing 10 units of I requires 5 of A, 5 of I and 8 of S. Producing 10 units of S requires 8 of A and 2 of S. Final demands are 20 in A and S, and 0 in I. Production is linear. (a) Write down the system of equations describing the economy. (b) Solve the system.

(a) Linearity lets us scale each recipe to *one* unit of output. Collect these *input coefficients* $t_{ij} =$ units of good i needed per unit of good j, column by column:

$$T = \begin{bmatrix} t_{AA} & t_{AI} & t_{AS} \\ t_{IA} & t_{II} & t_{IS} \\ t_{SA} & t_{SI} & t_{SS} \end{bmatrix} = \begin{bmatrix} 2/20 & 5/10 & 8/10 \\ 5/20 & 5/10 & 0 \\ 0 & 8/10 & 2/10 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.5 & 0.8 \\ 0.25 & 0.5 & 0 \\ 0 & 0.8 & 0.2 \end{bmatrix}.$$

Let x_A, x_I, x_S be gross outputs and $d = (20, 0, 20)'$ final demand. Market clearing for each good: *output = intermediate use (row of T times x) + final demand*:

Answer.

$$\begin{aligned} x_A &= 0.1x_A + 0.5x_I + 0.8x_S + 20 \\ x_I &= 0.25x_A + 0.5x_I \\ x_S &= 0.8x_I + 0.2x_S + 20 \end{aligned}$$

i.e. $x = Tx + d$, or in standard form $(I - T)x = d$:

$$\begin{bmatrix} 1 - 0.1 & -0.5 & -0.8 \\ -0.25 & 1 - 0.5 & 0 \\ 0 & -0.8 & 1 - 0.2 \end{bmatrix} \begin{bmatrix} x_A \\ x_I \\ x_S \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 20 \end{bmatrix}.$$

Note: Columns are *recipes* (what sector j buys); rows are *uses* (where good i goes). Once we have the per-unit receipes, we need to collect demand from all sectors of A and then for I and then S, those demands form the basis of the system of equations.

(b) Let $L = I - T$. Its determinant (Sarrus) shows it is invertible (does not collapse to lower dim space)

$$\begin{aligned}|L| &= 0.9(0.5)(0.8) + (-0.5)(0)(0) + (-0.8)(-0.25)(-0.8) \\ &\quad - (-0.8)(0.5)(0) - 0.9(0)(-0.8) - (-0.5)(-0.25)(0.8) = 0.36 - 0.16 - 0.1 = 0.1.\end{aligned}$$

Cofactor matrix and inverse:

$$C = \begin{bmatrix} +\begin{vmatrix} 0.5 & 0 \\ -0.8 & 0.8 \end{vmatrix} & -\begin{vmatrix} -0.25 & 0 \\ 0 & 0.8 \end{vmatrix} & +\begin{vmatrix} -0.25 & 0.5 \\ 0 & -0.8 \end{vmatrix} \\ -\begin{vmatrix} -0.5 & -0.8 \\ -0.8 & 0.8 \end{vmatrix} & +\begin{vmatrix} 0.9 & -0.8 \\ 0 & 0.8 \end{vmatrix} & -\begin{vmatrix} 0.9 & -0.5 \\ 0 & -0.8 \end{vmatrix} \\ +\begin{vmatrix} -0.5 & -0.8 \\ 0.5 & 0 \end{vmatrix} & -\begin{vmatrix} 0.9 & -0.8 \\ -0.25 & 0 \end{vmatrix} & +\begin{vmatrix} 0.9 & -0.5 \\ -0.25 & 0.5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.2 & 0.2 \\ 1.04 & 0.72 & 0.72 \\ 0.4 & 0.2 & 0.325 \end{bmatrix},$$

$$L^{-1} = \frac{C'}{|L|} = \frac{1}{0.1}\begin{bmatrix} 0.4 & 1.04 & 0.4 \\ 0.2 & 0.72 & 0.2 \\ 0.2 & 0.72 & 0.325 \end{bmatrix} = \begin{bmatrix} 4 & 10.4 & 4 \\ 2 & 7.2 & 2 \\ 2 & 7.2 & 3.25 \end{bmatrix}.$$

Hence

$$x = L^{-1}d = \begin{bmatrix} 4 & 10.4 & 4 \\ 2 & 7.2 & 2 \\ 2 & 7.2 & 3.25 \end{bmatrix}\begin{bmatrix} 20 \\ 0 \\ 20 \end{bmatrix} = \begin{bmatrix} 80+0+80 \\ 40+0+40 \\ 40+0+65 \end{bmatrix} = \begin{bmatrix} 160 \\ 80 \\ 105 \end{bmatrix}.$$

Answer. $x_A = 160$, $x_I = 80$, $x_S = 105$. (Cramer's rule should give you the same.)

Reading the Leontief inverse: direct + indirect requirements

Column j of $(I-T)^{-1}$ is the total output needed from every sector to deliver *one* extra unit of good j to final users. E.g. one more unit of services for consumers requires 4 units of A, 2 of I and 3.25 of S in total, far more than the direct recipe, because inputs need inputs, which need inputs, ... Formally, when the series converges,

$$(I-T)^{-1} = I + T + T^2 + T^3 + \cdots, \qquad x = \underbrace{d}_{\text{round 0}} + \underbrace{Td}_{\text{inputs for } d} + \underbrace{T^2 d}_{\text{inputs for those}} + \cdots$$

This is the input–output analogue, a production network multiplier of the Keynesian multiplier $1/(1-c) = 1 + c + c^2 + \cdots$ in Question 3. A shock to final demand for good j moves output by column j of the Leontief inverse: $\Delta\mathbf{x} = (I-T)^{-1}\Delta\mathbf{d}$, so $\partial x_i/\partial d_j = [(I-T)^{-1}]_{ij}$.

A one-unit shock to final demand for A raises the outputs of (A, I, S) by column 1 of the Leontief inverse, $\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$: a total output multiplier of 8 units of final output.

I has the largest multiplier ($10.4 + 7.2 + 7.2 = 24.8$) because its production is the most input-hungry: each unit of I uses 1.8 units of inputs, including 0.5 units of I itself, so its demand for I creates lots of chain-reaction demand for inputs, which feed back to more demand etc.

1 Appendix

A1) Where do the cofactors come from? Consider a 3×3: $A = [c_1|c_2|c_3]$. Write the first column as

$$c_1 = a_{11}e_1 + a_{21}e_2 + a_{31}e_3,$$

where $e_1 = (1, 0, 0)'$, $e_2 = (0, 1, 0)'$, $e_3 = (0, 0, 1)'$. A determinant is linear in each column (R1 and R2), so it splits into three pieces:

$$|A| = a_{11} \det[e_1, c_2, c_3] + a_{21} \det[e_2, c_2, c_3] + a_{31} \det[e_3, c_2, c_3].$$

Each piece reduces to a minor. Take the second: subtract a_{22} times column 1 from column 2, and a_{23} times column 1 from column 3. This clears row 2 without changing the determinant (R1 and R3). Then swap rows 1 and 2 to bring the 1 into the top-left corner; a swap flips the sign, since $\det[u, v] = -\det[v, u]$ (and rows behave like columns because $|A'| = |A|$):

$$\det[e_2, c_2, c_3] = \begin{vmatrix} 0 & a_{12} & a_{13} \\ 1 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & a_{12} & a_{13} \\ 1 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{12} & a_{13} \\ 0 & a_{32} & a_{33} \end{vmatrix} = -M_{21}.$$

The first piece needs no swap, giving $+M_{11}$. The third needs two adjacent swaps (row 3 past row 2, then past row 1), giving $+M_{31}$. Hence

$$detA = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}.$$

This method generalises to expanding any row or column, so look out for lots of 0s to make life easier.